Thin films of diblock copolymers: existence and two-dimensional regularity of minimizing configurations

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Abstract

We introduce a variational model for the study of thin films of diblock copolymers. We establish existence of minimizing configurations via relaxation of the energy involved. Moreover, we prove partial regularity results for mass-constrained minimizers in two space dimensions. To show this last result, we develop a two-dimensional regularity theory for partitions with quasi-minimal surface area, subject to an additional graph constraint.

1 Introduction

Block copolymers are an important class of soft materials (see [5]). They are composed by chemically bonded linear chains of monomers. The competition between the repulsion among different subchains and the entropy cost associated with chain stretching is the mechanism behind the extraordinary self-assembly property of block copolymers, that leads to the creation of fascinating patterns exhibiting interesting periodicity properties (see [36]). These beautiful examples of nature's regular shapes turn out to be extremely useful: indeed, block copolymers in the bulk are used in applications ranging from upholstery foam to box tape, from asphalt additives to drug delivery, from photonic crystals to nanoporous materials (see [19, 23]).

Models aimed at describing the behaviour of block copolymers from physics and chemistry can be roughly divided into two categories: (self consistent) mean fields models and density functional theory models. The former (see, for instance, [27, 26]) seem to give, at least numerically, predictions for the phase diagram that are quantitatively close to what is observed in experiments. Unfortunately, they are extremely complicated from the mathematical point of view and thus not suitable (at least so far) for rigorous analytical investigations. On the other hand, models within the framework of density functional theory are relatively more simple, while still capturing the main physics driving the arrangement of block copolymers.

A celebrated mean field model for block copolymers was derived by Ohta and Kawasaki in [30] for the case of diblock copolymers (two monomers) in the strong segregation regime by using several approximations (infinite temperature and thermodynamic limit). Despite this, it has successfully been used to derive qualitative properties related to both the dynamics

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and the statics of diblock copolymers. In mathematical terms, the Ohta-Kawasaki is a phase-field model given by the sum of a Cahn-Hilliard-type functional (replaced by a perimeter term in the sharp-interface version) and a nonlocal interaction term. The former models the short-range interaction between different monomers, related to the surface energy of the interfaces dividing the regions of high concentration of the two monomer species, while the latter represents their long-range interaction. The emergence of highly nontrivial pattern configurations at a mesoscopic scale is precisely due to the competition between these two kinds of energies.

From the mathematical point of view, the interest in studying the Ohta-Kawasaki model, and variants of it, comes from the many fascinating and extremely challenging questions that it inspires, and that can be tackled by using rigorous analytical tools. For these reasons, it has attracted large attention from the mathematical community over the years, and it is still a very active area of research. In particular, an important open problem is to rigorously prove, in any spatial dimension, the periodicity of minimizers (up to boundary effects). The closer results in this direction are those of Müller [29] in one dimension (see also [32, 33, 40]), and of Alberti, Choksi, and Otto [2] in general dimension.

When the copolymers are constrained in a thin film, surface energies, namely the interactions of the copolymers with the substrate and the air, strongly influence the shape of the patterns that are formed, making the landscape of configuration drastically different from that of the bulk case. The possibility of accessing a larger class of equilibrium configurations has been exploited for many applications (see [34]). Patterns in thin films of block copolymers have been investigated numerically (see, for instance, [18, 21, 24, 25, 31, 35]), but despite the large interest in the physical community in understanding this phenomenon, to the best of our knowledge, no analytical rigorous study on thin films of block copolymers is available.

In this paper we lay the foundation for the analytical investigation of thin films of diblock copolymers by introducing a variational model where the copolymer film is confined between a solid substrate on one side, and the other surface is exposed. The interface between the copolymer phase and the region above it (which could be void, air or a liquid solvent) corresponds to a free surface. The region above the film is model by a homopolymer, in the framework of the density functional theory for blends of diblock copolymers with homopolymers derived by Choksi and Ren [10] (see also [6, 37, 38] for related studies in the mathematical literature). The copolymer film is mathematically described by the subgraph of a function; in this regards, the model is also reminiscent and partially inspired by variational models for epitaxially strained elastic films, see [7, 9, 14, 12, 11].

We start a rigorous investigation of the properties of the underlying energy and of the equilibrium configurations of the system. In particular, we discuss the lower semicontinuity properties of the energy, which permits to prove existence of minimizing configurations via relaxation. Moreover, we establish several regularity properties of minimizers in dimension two. Further investigations on the fine structure of patterns in thin films of diblock copolymers will be the subject of future work.

1.1 Main results

We now pass to an introductory description of the model and of the main results obtained in this paper. For the precise definitions and assumptions we refer to Section 2.

We consider a thin film of diblock copolymers in general dimension $n \geq 2$, whose configuration is described by a phase variable u taking values $+1$, -1, and 0, representing the two

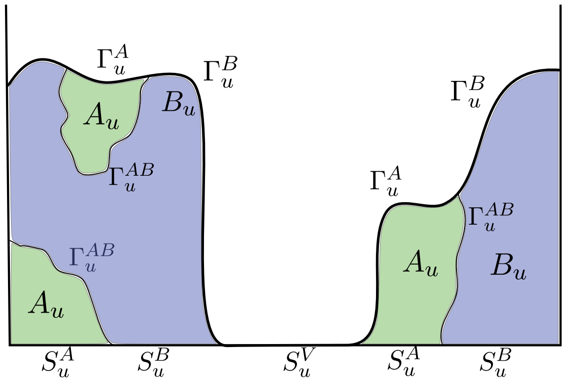

Figure 1: The different phases and interfaces of an admissible configuration.

phases $A_u = \{u = 1\}$ (first monomer), $B_u = \{u = -1\}$ (second monomer) and the region above the film $V_u = \{u = 0\}$. Admissible configurations are those for which the region $A_u \cup B_u$ occupied by the diblock copolymer is confined in the subgraph of a function h_u over the flat substrate (assumed to be infinitely large with respect to the film), whose graph Γ represents the *free profile* of the film (see Figure 1). As customary in this kind of problems, to focus on the effect of the surface energy on the equilibrium configurations, we work with lateral periodic boundary conditions.

We consider a sharp-interface model in which the short-range interaction energy $\mathscr{G}(u)$ of a configuration u is assumed to be proportional to the surface measure of the interfaces between the different phases, with possibly different surface tensions. The interfaces involved are: Γ_u^{AB} (between the two phases of the diblock copolymer inside the film), Γ_u^A, Γ_u^B (between each polymer and the air), and, since also the contact between the film and the substrate costs surface energy, S_u^A, S_u^B (between each polymer and the substrate), S_u^V (between the substrate and the air).

The long-range interaction $\mathcal{N}(u)$ (see Section 2.3 for the precise definition) describes the repulsion force between different monomers, and thus acts only on the two sets A_u and B_u, while the air is modeled as a homopolymer, whose only interaction with the diblock copolymer is via the surface energies above. We stress that, for the results contained in this paper, the precise form of the nonlocal energy does not play a role, and the only property that we use is that $\mathcal{N}(\cdot)$ is Lipschitz continuous with respect to the symmetric difference of sets, see Proposition 2.2. However, the explicit expression of $\mathcal{N}$ will be crucial for the characterization of optimal or equilibrium configurations.

The total energy $\mathscr{F}(u)$ of a regular configuration u, whose profile is given by a Lipschitz function h_u, writes as

$$\begin{aligned}\mathscr{F}(u) &:= \mathscr{G}(u) + \gamma\mathcal{N}(u)\\ &:= \sigma_A\mathcal{H}^{n-1}(\Gamma_u^A) + \sigma_B\mathcal{H}^{n-1}(\Gamma_u^B) + \sigma_{AB}\mathcal{H}^{n-1}(\Gamma_u^{AB})\\ &\quad + \sigma_{AS}\mathcal{H}^{n-1}(S_u^A) + \sigma_{BS}\mathcal{H}^{n-1}(S_u^B) + \sigma_S\mathcal{H}^{n-1}(S_u^V) + \gamma\mathcal{N}(u)\end{aligned}$$

(see Section 2 for the precise definition of all the terms involved).

The rest of the paper is divided in two parts, where we study several properties of the energy $\mathscr{F}$. In the first part we focus on its lower semicontinuity with respect to the L^1-topology, and we identify in Theorem 3.1 the lower semicontinuous envelope $\overline{\mathscr{F}}$, defined over a larger class of possibly irregular profiles. In particular, the relaxation procedure allows to consider configurations whose free boundary is described by a function h_u of bounded variation: it then might be unbounded and with jump discontinuities, thus allowing to describe configurations observed in experiments, like steps.

The functional $\overline{\mathscr{F}}$ has the same form of the original functional $\mathscr{F}$, namely it is the sum of a surface energy contribution $\overline{\mathscr{G}}(u)$ and of the nonlocal interaction $\gamma\mathcal{N}(u)$. Notice that the relaxation affects only the surface part of the energy, as the nonlocal term is continuous with respect to L^1-convergence. As might be expected, the new surface energy $\overline{\mathscr{G}}$ has relaxed surface tension coefficients, due to the possibility of reducing the energy by inserting a thin layer of a phase between two other phases (wetting). The non standard aspect of this procedure is that, due to the additional constraints of the model (namely, the only admissible configurations are subgraphs), not all the possible infiltrations are allowed; this prevents us to apply directly the well-known results about the relaxation of surface energy of clusters in $\mathbb{R}^n$, see [3].

Concerning the proof, whereas the liminf inequality follows by a standard argument adapted to our setting (see Proposition 3.8), the construction of a recovery sequence (Proposition 3.9) requires extra care. Indeed, once a non-regular profile is approximated by a Lipschitz one, thanks to a construction by Chambolle and Solci [9], we need to carefully insert thin layers of a phase in between the profile and the substrate, the profile and the air, and the air and the substrate, by preserving the graph constraint, as well as the mass constraint.

Finally, the existence of a solution to the mass constrained minimization problem for the relaxed functional

$$\min\{\overline{\mathscr{F}}(u) \,:\, |A_u| + |B_u| = M,\, |A_u| = m\}, \tag{1.1}$$

where $0 < m < M$, follows by a standard application of the direct method (see Theorem 3.5).

In the second part of the paper we turn our attention to the study of regularity properties of solutions to (1.1). This is where the main mathematical challenges are, stemming from the fact that admissible competitors have to satisfy the additional condition of being subgraphs. Indeed, if no graph constraint is in force, then partial regularity of minimizing clusters could be obtained by a standard strategy, which would amount to first showing that volume-constrained minimizers are quasi-minimizers of the surface energy, and then to proving an elimination property (see [20]) which allows to reduce locally to the case of only two interfaces. Once this is done, partial regularity follows from classical results (see [17]). In our case, though, we cannot apply directly those results, as they require to make *arbitrary* perturbations, thus possibly exiting the restricted class of admissible configurations. For this reason, we need to perform delicate geometric constructions, and to combine several ideas in order to prove regularity.

We next summarize our main strategy. In Lemma 4.2 we remove the mass constraints by showing that every solution to (1.1) is also a solution to a suitable penalized problem. The proof of this fact follows a rather standard contradiction argument, which amounts to show that if a minimizer of the penalized problem does not satisfy the volume constraint, then it is possible to modify it and reduce its energy - which would be a contradiction - provided that the constant in front of the penalization term is large enough. When there is just one mass constraint and the problem is in the whole space $\mathbb{R}^n$, this can be achieved by a suitable rescaling of the minimizer; we follow here a refined idea of Esposito and Fusco [13], who showed that the same can be done by a local perturbation of the set, which brings the mass of the perturbed set closer (but not necessarily equal) to the desired mass. However, a main difference is that, while in [13] the local variation is *radial*, in our case we will perform a *vertical* local rescaling in order to keep the graph constraint. Furthermore, another difference is that in our case two mass constraints are in force; we can however avoid the use of the Implicit Function Theorem (used in arguments like that in [22, Lemma 29.14]) and deal with the two constraints one at a time.

The penalized minimum problem allows us to work with a larger class of perturbations. This fact, together with the Lipschitz continuity of the nonlocal energy, immediately implies (see Proposition 4.3) that every solution u to (1.1) is a quasi-minimizer of the surface energy $\overline{\mathscr{G}}$, in the sense that there exists $\Lambda > 0$ such that

$$\overline{\mathscr{G}}(u) \leq \overline{\mathscr{G}}(v) + \Lambda\big(|A_u \triangle A_v| + |B_u \triangle B_v|\big), \tag{1.2}$$

for all admissible competitors v. Notice that in this formulation, admissible competitors have still to satisfy the graph constraint, and thus the regularity does not follow from classical results. We denote by $\mathcal{A}_{\Lambda,M}$ the class of quasi-minimizers satisfying the inequality (1.2) and with total mass M, see Definition 4.1. By using (1.2) we then show that h_u is bounded, see Proposition 4.4.

The next main result, which is proved in Subsection 4.2 through a series of propositions, concerns the regularity of quasi-minimizers in dimension $n = 2$. In view of the previous discussion, it applies in particular to any solution of the minimum problem (1.1).

Theorem 1.1 (Partial regularity in dimension $n = 2$). *Assume that $n = 2$, and that the surface tension coefficients satisfy the strict triangle inequalities*

$$\sigma_{AB} < \sigma_A + \sigma_B, \qquad \sigma_A < \sigma_B + \sigma_{AB}, \qquad \sigma_B < \sigma_A + \sigma_{AB}. \tag{1.3}$$

Let $u \in \mathcal{A}_{\Lambda,M}$ be a quasi-minimizer, according to Definition 4.1. Then the followings hold.

(i) (Infiltration) *There exists $\varepsilon_0 > 0$ (depending only on M, Λ, and the surface energy coefficients) such that, for any square $\mathcal{Q}_r(z_0)$ centered at $z_0 \in \mathbb{R}^n$ with side length $r \in (0,1)$, the following implications hold:*

$$|V_u \cap \mathcal{Q}_r(z_0)| < \varepsilon_0 r^2 \qquad \Rightarrow \qquad |V_u \cap \mathcal{Q}_{\frac{r}{2}}(z_0)| = 0,$$

and, if $\mathcal{Q}_r(z_0)$ does not intersect the substrate,

$$|(A_u \cup B_u) \cap \mathcal{Q}_r(z_0)| < \varepsilon_0 r^2 \qquad \Rightarrow \qquad |(A_u \cup B_u) \cap \mathcal{Q}_{\frac{r}{2}}(z_0)| = 0.$$

(ii) (Lipschitz regularity of the graph) *There exists a finite set Σ, containing the jump points of h_u, such that h_u is locally Lipschitz outside Σ.*

(iii) (Singular set) *At the upper end of a jump point of h_u, the graph has a vertical tangent. At the points of Σ that are not jump points of h_u, the left or the right derivative of h_u is infinite. The graph of h_u does not contain interior or exterior cusps.*

(iv) (Internal regularity of Γ_u^{AB}) *For every $\alpha \in (0, 1/2)$ the interface $\partial A \cap \partial B$ is a locally a $C^{1,\alpha}$-curve in $\{(x,y) \in \mathbb{R}^2 : 0 < y < h_u(x)\}$.*

(v) ($C^{1,\alpha}$-regularity of the graph) *If $x_0 \notin \Sigma$ is such that $(x_0, h_u(x_0)) \in \partial^* A \cup \partial^* B$, then h_u is of class $C^{1,\alpha}$ in a neighbourhood of x_0, for every $\alpha \in (0, 1/2)$.*

Conditions (1.3) are known to be needed in order to get regularity for minimizing clusters (see [20, 39]). The elimination property is well-known in the case of minimal clusters (see [20]). The idea of the proof is to construct a suitable competitor by *filling* the minority phase in $\mathcal{Q}_r(z_0)$ with one of the other phases. However, in our case filling A_u or B_u by V_u might lead to a configuration which violates the graph constraint. Therefore, the proof of the infiltration for V_u (Proposition 4.6) and for $A_u \cup B_u$ (Proposition 4.7) uses a two step strategy: first, we prove the elimination property in a semi-infinite strip, where it is possible to fill $A_u \cup B_u$ with V_u, without violating the graph constraint; then, we show that a minimal configuration having small volume percentage of the void (or of the subgraph) in a cube must necessarily have a small volume percentage of the same in the semi-infinite strip, so that it is possible to conclude by using the first step.

The proof of the Lipschitz regularity follows an idea by Chambolle and Larsen [8] (see also [14, 15]): we show an *interior ball condition* (see Proposition 4.9), namely that there exists a uniform radius $\rho_0 > 0$ such that, for each z on the graph of h_u, it is possible to find a ball with radius ρ_0 tangent to the graph of h_u only at the point z and contained in the subgraph of h_u. This property implies (Proposition 4.10) that h_u has only a finite number of jump points, and that h_u is locally Lipschitz continuous outside a finite set (where the inner ball is tangent to the graph horizontally).

Since in two dimensions the graph h_u is closed, for each point z on the internal interface between the two phases it is possible to find a ball centered at z that does not intersect the graph, nor the substrate. Therefore, since internal interfaces do no have any graph constraint to satisfy, their $C^{1,\alpha}$-regularity follows from classical results (see Remark 4.12).

Finally, the proof of the $C^{1,\alpha}$ regularity of the graph (Proposition 4.13) is also based on an elimination property for the two sets A_u, B_u separately. To obtain this, we observe that thanks to the Lipschitz regularity of h_u, for every point $(x_0, h_u(x_0)) \in \partial^* A \cup \partial^* B$ with $x_0 \notin \Sigma$ we can find a rectangle such that the graph of h_u does not intersect its upper and lower sides. This property allows to perform a local perturbation which preserves the graph constraint.

We conclude this introduction with a few more remarks. As already observed, for what concerns the nonlocal part of the energy, our arguments rely only on its Lipschitz continuity and not on its explicit expression, so that all the results would continue to hold for any other energy term satisfying the same property.

The extension to the case of more than two phases is relatively straightforward and the arguments presented here can be directly generalized, at the price of a more demanding notation and of a larger number of different cases to be taken into consideration. It could also be possible to extend our results to different kinds of boundary conditions, or if surface interactions with horizontal walls are presents.

Finally, we believe that the two-dimensional regularity theory for quasi-minimal partitions subject to a graph constraint, developed in Section 4.2, has its own interest, besides the specific application to the diblock copolymer model. The generalization to higher dimensions is, however, not straightforward and would require new ideas: while we believe that the elimination property might be obtained by refined but similar arguments, the inner ball condition leading to the Lipschitz regularity of the graph is a purely two dimensional strategy. This, together with the study of finer regularity properties in two dimension, as well as the characterization of optimal configurations, will be the object of future research.

Structure of the paper. The paper is organized as follows. In Section 2 we introduce the main notation, the class of admissible configurations and the total energy of the system.

In Section 3 we compute the relaxation of the energy (Theorem 3.1) and we use this result to prove the existence of minimizing configurations (Theorem 3.5). In Section 4 we first show that solutions to the minimum problem (1.1) are quasi-minimizers of the surface energy under a graph constraint (Subsection 4.1), and then we prove Theorem 1.1 on the regularity of quasi-minimizers in dimension two (Subsection 4.2).

2 The model

2.1 Notation for functions of bounded variation and perimeters

The profile of the film will be modeled by the (generalized) graph of a periodic function with finite total variation in $(0, L)^{n-1}$ $(n \geq 2)$, where $L > 0$ is a fixed parameter, and its subgraph will represent the reference configuration of the film. We therefore firstly recall a few notions from the theory of BV-functions (see [4]), in order to fix the notation used in the paper. Given $h \in L^1_{\rm loc}(\Omega)$, where $\Omega \subset \mathbb{R}^m$ is an open set $(m \geq 1)$, its total variation is defined as

$$|Dh|(\Omega) \coloneqq \sup\left\{ \int_\Omega h \,\mathrm{div}\phi \,\mathrm{d}x \,:\, \phi \in C^\infty_{\rm c}(\Omega;\mathbb{R}^m),\, |\phi| \leq 1 \right\},$$

and this quantity is finite if and only if the distributional derivative Dh of h is a bounded Radon measure on Ω. We let $\mathrm{BV}(\Omega) \coloneqq \{h \in L^1(\Omega) \,:\, |Dh|(\Omega) < \infty\}$. If $h \in \mathrm{BV}(\Omega)$, at each point $x \in \Omega$ the approximate upper and lower limits

$$\begin{aligned} h^+(x) &\coloneqq \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{\rho\to 0} \frac{\mathscr{L}^m(\{h > t\} \cap B_\rho(x))}{\omega_m \rho^m} = 0 \right\}, \\ h^-(x) &\coloneqq \sup\left\{ t \in \mathbb{R} \,:\, \limsup_{\rho\to 0} \frac{\mathscr{L}^m(\{h < t\} \cap B_\rho(x))}{\omega_m \rho^m} = 0 \right\} \end{aligned} \tag{2.1}$$

are well-defined, where $\mathscr{L}^m$ is the m-dimensional Lebesgue measure, $B_\rho(x) \subset \mathbb{R}^m$ is the ball centered at x with radius ρ, and $\omega_m = \mathscr{L}^m(B_1(0))$. The *jump set* of h is then defined as the set

$$J_h \coloneqq \{x \in \Omega \,:\, h^-(x) < h^+(x)\}, \tag{2.2}$$

and it is well-known that J_h is a $(\mathcal{H}^{m-1}, m-1)$ rectifiable set, with normal $\nu_h(x)$ at $\mathcal{H}^{m-1}$-a.e. point $x \in J_h$.

We also recall that a set $E \subset \Omega$ has finite perimeter in Ω if $|D\chi_E|(\Omega) < \infty$, where $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if $x \notin E$; the *perimeter* of E in Ω is then defined as

$$\mathscr{P}(E;\Omega) \coloneqq |D\chi_E|(\Omega). \tag{2.3}$$

We introduce the *essential boundary* of E

$$\partial_e E \coloneqq \Omega \backslash (E^0 \cup E^1), \tag{2.4}$$

where, for $t \in [0, 1]$, E^t denotes the set of points where E has Lebesgue density t. Another relevant subset of the boundary of a set of finite perimeter is the *reduced boundary* $\partial^* E$ (see [4]). At every point of the reduced boundary the measure-theoretic outer normal ν_E is defined, the Lebesgue density of E is equal to $1/2$, and it is well-known that $\partial_e E$ coincides with $\partial^* E$ up to a $\mathcal{H}^{m-1}$-negligible set. We finally recall that a *Caccioppoli partition* of Ω is a finite partition $\{E_i\}_{i\in\{1,\dots,N\}}$ of Ω, $N \in \mathbb{N}$, such that $\sum_{i=1}^N \mathscr{P}(E_i;\Omega) < +\infty$. For a Caccioppoli partition $\{E_i\}_i$, $\mathcal{H}^{m-1}$-a.e. point of Ω belongs to one of the sets $(E_i)^1$ or to one of the intersections $\partial^* E_i \cap \partial^* E_j$ $(i \neq j)$.

2.2 Admissible configurations

We now describe the class of admissible configurations. Throughout the paper, we will denote by $x = (x', x_n)$ the generic point in $\mathbb{R}^n \equiv \mathbb{R}^{n-1}\times\mathbb{R}$, and by $\mathbb{R}^n_+ \coloneqq \mathbb{R}^{n-1}\times[0,\infty)$. The canonical basis of $\mathbb{R}^n$ will be denoted by $(e_1,\dots,e_n)$, and the Lebesgue measure on $\mathbb{R}^n$ by $|\cdot| \coloneqq \mathscr{L}^n(\cdot)$. Given $L>0$, we also set

$$Q_L \coloneqq [0,L)^{n-1} \subset \mathbb{R}^{n-1}, \qquad Q_L^+ \coloneqq Q_L \times [0,+\infty). \tag{2.5}$$

We assume that the substrate occupies the infinite region

$$S \coloneqq \mathbb{R}^{n-1} \times (-\infty, 0). \tag{2.6}$$

We introduce the class of *admissible profiles*

$$\mathcal{AP}(Q_L) \coloneqq \Big\{ h : \mathbb{R}^{n-1} \to [0,+\infty) \,:\, h \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n-1}),\, h \text{ is } Q_L\text{-periodic} \Big\}. \tag{2.7}$$

The reference configuration of the film is represented by the subgraph of an admissible profile $h \in \mathcal{AP}(Q_L)$: we denote it and its periodic extension by

$$\begin{aligned} \Omega_h &\coloneqq \Big\{ (x', x_n) \in Q_L \times \mathbb{R} \,:\, 0 < x_n < h(x') \Big\}, \\ \Omega_h^\# &\coloneqq \Big\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, 0 < x_n < h(x') \Big\}, \end{aligned} \tag{2.8}$$

respectively. Notice that, as h has finite total variation, the set Ω_h has finite perimeter. We also define, for $h \in \mathcal{AP}(Q_L)$, the *free profile*

$$\Gamma_h \coloneqq \Big\{ (x', x_n) \,:\, x' \in Q_L,\, h^-(x') \le x_n \le h^+(x') \Big\}, \tag{2.9}$$

and we denote by $\Gamma_h^\#$ its periodic extension. Notice that if $0 < x_n < h^-(x')$ then $x \in (\Omega_h^\#)^1$, while if $x_n > h^+(x')$ then $x \in (\Omega_h^\#)^0$; therefore $\partial_e(\Omega_h^\# \cup S)$ is a subset of $\Gamma_h^\#$ (and coincides with $\Gamma_h^\#$ up to a $\mathcal{H}^{n-1}$-negligible set).

The region Ω_h occupied by the film is filled with a diblock copolymer, that is, we have a partition of Ω_h into two disjoint sets of finite perimeter A, B representing the two phases of the diblock copolymer. We identify these two phases with the level sets of a marker function $u : \Omega_h \to \{\pm 1\}$ with bounded variation, so that $A = \{u = 1\}$ and $B = \{u = -1\}$. As Ω_h is in general not an open set, it will be convenient to consider u as a piecewise constant function defined in the full space $\mathbb{R}^n$, taking two additional values $u = 0$ and $u = 2$ in the region above the film and in the substrate, respectively. This is made precise by the following definition.

Definition 2.1 (Admissible configurations)**.** *Let $I \coloneqq \{\pm 1, 0, 2\}$. The class $\mathcal{X}$ of* admissible configurations *is the space of functions $u : \mathbb{R}^n \to I$ satisfying the following properties:*

(i) $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n; I)$,

(ii) $u(x' + Le_i, x_n) = u(x', x_n)$ *for all* $(x', x_n) \in \mathbb{R}^n$, $i = 1, \dots, n-1$,

(iii) there exists $h_u \in \mathcal{AP}(Q_L)$ *such that* $\Omega_{h_u}^\# = \{u = 1\} \cup \{u = -1\}$,

(iv) $S = \{u = 2\}$, *where* S *is the substrate defined in* (2.6)

(the previous identities have to be understood in the almost everywhere sense with respect to $\mathscr{L}^n$*). The class of* regular admissible configurations *is defined as*

$$\mathcal{X}_{\mathrm{reg}} := \Big\{ u \in \mathcal{X} \,:\, h_u \textit{ is Lipschitz continuous} \Big\}. \tag{2.10}$$

We consider the space $\mathcal{X}$ *endowed with the* L^1*-convergence: we say that a sequence* $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{X}$ *converges in* $\mathcal{X}$ *to* $u \in \mathcal{X}$ *if* $u_k \to u$ *in* $L^1(Q_L^+)$*.*

Given an admissible configuration $u \in \mathcal{X}$, we have a partition of the strip Q_L^+ into three sets of finite perimeter, which will be denoted by

$$A_u := \{u = 1\} \cap Q_L^+, \qquad B_u := \{u = -1\} \cap Q_L^+, \qquad V_u := \{u = 0\} \cap Q_L^+, \tag{2.11}$$

and as usual we will denote by $A_u^\#$, $B_u^\#$ and $V_u^\#$ their periodic extensions. The sets A_u and B_u represent the two phases occupied by the diblock copolymer, and V_u represents the void (or homopolymer) above the film. Notice that $A_u \cup B_u = \Omega_{h_u}$ and $\partial^* V_u^\# = \Gamma_{h_u}^\#$ (up to a $\mathcal{H}^{n-1}$-negligible set). In other words, the admissible configurations are just periodic partitions of the upper half-space into three sets of locally finite perimeter A, B, V, with the constraint that $A \cup B$ is the subgraph of a BV-function. The jump set J_u of u coincides (up to a $\mathcal{H}^{n-1}$-negligible set) with the union of their reduced boundaries:

$$\mathcal{H}^{n-1}\bigg(J_u \setminus \bigcup_{\substack{i,j\in I \\ i\neq j}} \partial^*\{u = i\} \cap \partial^*\{u = j\} \bigg) = 0, \tag{2.12}$$

with $(u^+(x), u^-(x)) = (i, j)$ (up to a permutation) for every $x \in \partial^*\{u = i\} \cap \partial^*\{u = j\}$.

As we want to consider different values of the surface tension for all the possible different interfaces between the phases, it is convenient to introduce the following notation (see Figure 1):

$$\Gamma_u^A := \Gamma_{h_u} \cap \partial^* A_u^\#\,, \qquad \Gamma_u^B := \Gamma_{h_u} \cap \partial^* B_u^\#\,, \qquad \Gamma_u^{AB} := \partial^* A_u^\# \cap \partial^* B_u^\# \cap Q_L^+\,, \tag{2.13}$$

and

$$S_u^A := \partial^* A_u \cap \big(Q_L \times \{0\}\big), \quad S_u^B := \partial^* B_u \cap \big(Q_L \times \{0\}\big), \quad S_u^V := \Gamma_{h_u} \cap \big(Q_L \times \{0\}\big). \tag{2.14}$$

The set S_u^V represents the possible region in which the substrate is exposed. In view of (2.12), the disjoint union of these interfaces coincides with the jump set J_u of u inside the periodicity strip:

$$J_u \cap Q_L^+ = \Gamma_u^A \cup \Gamma_u^B \cup \Gamma_u^{AB} \cup S_u^A \cup S_u^B \cup S_u^V \cup N \tag{2.15}$$

with $\mathcal{H}^{n-1}(N) = 0$.

2.3 The nonlocal energy

We next introduce, following [10] and modeling the phase V_u as an homopolymer, the nonlocal interaction energy between the two phases A_u, B_u of an admissible configuration $u \in \mathcal{X}$.

For $u \in \mathcal{X}$ we let $\bar{u} := \int_{Q_L^+} u(x)\,\mathrm{d}x = |A_u| - |B_u|$, and we define

$$\mathcal{N}(u) := \int_{Q_L^+} |\nabla \phi_u(x)|^2 \,\mathrm{d}x\,, \tag{2.16}$$

where the potential $\phi_u : Q_L^+ \to \mathbb{R}$ associated to the configuration $u \in \mathcal{X}$ is the solution to

$$-\Delta\phi_u = u - \bar{u} \quad \text{in } Q_L^+, \qquad\qquad \int_{Q_L^+} \phi_u(x)\,\mathrm{d}x = 0,$$

with periodic boundary conditions on the lateral boundary $\partial Q_L \times (0, +\infty)$ and zero Neumann boundary condition at the interface $Q_L \times \{0\}$ with the substrate.

The only property about the nonlocal energy that we will use for the results in this paper is its Lipschitz continuity with respect to the L^1-distance of the phases.

Proposition 2.2. *Given a positive constant $M > 0$, there exists $L_{\mathcal{N}} \in (0, +\infty)$, depending on M, such that for all $u, v \in \mathcal{X}$ with $|\Omega_{h_u}|, |\Omega_{h_v}| \le M$ it holds*

$$|\mathcal{N}(u) - \mathcal{N}(v)| \le L_{\mathcal{N}}\left(|A_u \triangle A_v| + |B_u \triangle B_v| \right).$$

Proof. The result follows by arguing as in [1, Lemma 2.6]. □

2.4 The energy of regular configurations

We now introduce the energy associated with a regular configuration $u \in \mathcal{X}_{\mathrm{reg}}$. This energy will be extended to the whole space $\mathcal{X}$ of admissible configurations in Section 3 via a relaxation procedure. The total energy is the sum of the nonlocal energy $\mathcal{N}(u)$, defined in (2.16), and a penalization of the interfaces between the phases.

Definition 2.3 (Energy)**.** *Given positive coefficients $\sigma_A, \sigma_B, \sigma_{AB}, \sigma_{AS}, \sigma_{BS}, \sigma_S, \gamma > 0$, we define the total energy of a regular configuration $u \in \mathcal{X}_{\mathrm{reg}}$ as*

$$\begin{aligned} \mathscr{F}(u) := {} & \sigma_A \mathcal{H}^{n-1}(\Gamma_u^A) + \sigma_B \mathcal{H}^{n-1}(\Gamma_u^B) + \sigma_{AB} \mathcal{H}^{n-1}(\Gamma_u^{AB}) + \gamma \mathcal{N}(u) \\ & + \sigma_{AS} \mathcal{H}^{n-1}(S_u^A) + \sigma_{BS} \mathcal{H}^{n-1}(S_u^B) + \sigma_S \mathcal{H}^{n-1}(S_u^V)\,. \end{aligned} \tag{2.17}$$

By introducing the surface energy density

$$\Psi(i,j) := \begin{cases} \sigma_A & \text{if } (i,j) = (1,0), \\ \sigma_B & \text{if } (i,j) = (-1,0), \\ \sigma_{AB} & \text{if } (i,j) = (1,-1), \\ \sigma_{AS} & \text{if } (i,j) = (1,2), \\ \sigma_{BS} & \text{if } (i,j) = (-1,2), \\ \sigma_S & \text{if } (i,j) = (0,2), \end{cases} \tag{2.18}$$

with $\Psi(i,j) = \Psi(j,i)$, we can write in a more compact notation an equivalent representation of the energy in terms of the jump set of the piecewise constant function u (see (2.15)):

$$\mathscr{F}(u) = \int_{J_u \cap Q_L^+} \Psi(u^+, u^-)\,\mathrm{d}\mathcal{H}^{n-1} + \gamma \mathcal{N}(u). \tag{2.19}$$

3 Relaxation and existence of minimizers

The goal of this section is to compute the lower semicontinuous envelope $\overline{\mathscr{F}}$ of the functional $\mathscr{F}$ with respect to the convergence in $\mathcal{X}$, under a volume constraint: for every $u \in \mathcal{X}$

$$\overline{\mathscr{F}}(u) := \inf\left\{\liminf_{k\to\infty} \mathscr{F}(u_k) \;:\; u_k \in \mathcal{X}_{\text{reg}},\, |A_{u_k}| = |A_u|,\, |B_{u_k}| = |B_u|,\, u_k \to u \text{ in } \mathcal{X}\right\}. \quad (3.1)$$

In the following theorem, which is proved in Subsections 3.1 and 3.2, we give a representation formula for the relaxed functional $\overline{\mathscr{F}}$.

Theorem 3.1 (Relaxation). *Assume that $\sigma_{AB} \le \sigma_A + \sigma_B$. Then the functional $\overline{\mathscr{F}}$, defined in (3.1), is given by*

$$\overline{\mathscr{F}}(u) = \int_{J_u \cap Q_L^+} \overline{\Psi}(u^+, u^-)\, \mathrm{d}\mathcal{H}^{n-1} + \gamma \mathcal{N}(u) \quad (3.2)$$

for all $u \in \mathcal{X}$, where

$$\overline{\Psi}(i,j) := \begin{cases} \bar{\sigma}_A := \min\{\sigma_A, \sigma_B + \sigma_{AB}\} & \text{if } (i,j) = (1,0), \\ \bar{\sigma}_B := \min\{\sigma_B, \sigma_A + \sigma_{AB}\} & \text{if } (i,j) = (-1,0), \\ \sigma_{AB} & \text{if } (i,j) = (1,-1), \\ \min\{\sigma_{AS}, \sigma_{BS} + \sigma_{AB}\} & \text{if } (i,j) = (1,2), \\ \min\{\sigma_{BS}, \sigma_{AS} + \sigma_{AB}\} & \text{if } (i,j) = (-1,2), \\ \min\{\sigma_S, \sigma_{AS} + \bar{\sigma}_A, \sigma_{BS} + \bar{\sigma}_B\} & \text{if } (i,j) = (0,2), \end{cases} \quad (3.3)$$

and $\overline{\Psi}(i,j) = \overline{\Psi}(j,i)$.

Remark 3.2. *From the proof of Theorem 3.1, it also follows that the representation formula (3.2) continues to hold if we drop the mass constraints in the definition (3.1) of $\overline{\mathscr{F}}$.*

Remark 3.3. *The assumption $\sigma_{AB} \le \sigma_A + \sigma_B$ prevents the possibility of reducing the energy by inserting of a thin layer of void between the phases A and B, and is justified by the fact that the subchains of type A and B of a diblock copolymer are chemically bonded together. In case the opposite inequality holds, the relaxed functional would have a different surface tension ($\sigma_A + \sigma_B$) only for the vertical interfaces of Γ^{AB} connected to the graph.*

Remark 3.4. *The choice of the L^1 topology is justified by the fact that we do not consider elastic effects, that would lead to cracks inside the copolymer phases. In case these effects have to be taken into account, a natural topology would be the Hausdorff convergence of the epigraph of the profile, as in [7, 14]; the corresponding relaxed functional would contain additional terms accounting for vertical cracks, connected to the free profile of the film, inside the two phases of the copolymer.*

The existence of minimizers of the relaxed functional $\overline{\mathscr{F}}$ follows by a standard application of the direct method of the Calculus of Variations. We fix two positive real numbers $M > 0$ and $m \in (0, M)$, which represent the total volume of the film and the volume of the phase A of the copolymer, respectively.

Theorem 3.5 (Existence of minimizers)**.** *Under the assumptions of Theorem 3.1, the constrained minimization problem*

$$\min\{\overline{\mathscr{F}}(u) : u \in \mathcal{X}, |A_u| = m, |B_u| = M - m\} \tag{3.4}$$

admits a solution. Furthermore, if $\bar{u} \in \mathcal{X}$ is a solution of the above problem, then

$$\overline{\mathscr{F}}(\bar{u}) = \inf\{\mathscr{F}(u) : u \in \mathcal{X}_{\mathrm{reg}}, |A_u| = m, |B_u| = M - m\}. \tag{3.5}$$

Proof. Let $\{u_k\}_k \subset \mathcal{X}$, with $|A_{u_k}| = m$, $|B_{u_k}| = M - m$, be a minimizing sequence for the minimum problem (3.4). Consider first the profiles $h_k := h_{u_k} \in \mathcal{AP}(Q_L)$. The equiboundedness of the energies $\sup_k \overline{\mathscr{F}}(u_k) < \infty$, together with the constraint $\|h_k\|_{L^1(Q_L)} = |\Omega_{h_k}| = M$ and the periodicity of h, yields that the sequence $\{h_k\}_k$ is uniformly bounded in $\mathrm{BV}(K)$ for every compact set $K \subset \mathbb{R}^{n-1}$. Therefore up to (not relabeled) subsequences we have that $h_k \to h$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$, for some limit admissible profile $h \in \mathcal{AP}(Q_L)$ such that $|\Omega_h| = M$.

Next, consider the sequence $\{u_k\}_k$. Again by equiboundedness of the energies we have $\sup_k |Du_k|(Q_L^+) < \infty$, therefore (using also the periodicity) $u_k \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and almost everywhere, for some periodic function $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n; I)$. It follows that

$$\{u = \pm 1\} = \lim_k \{u_k = \pm 1\} = \lim_k \Omega_{h_k} = \Omega_h,$$

which shows that $h = h_u$. Therefore u is an admissible configuration in the sense of Definition 2.1. Moreover, by L^1-convergence u satisfies the constraint $|A_u| = m$, $|B_u| = M - m$.

By lower semicontinuity of the relaxed functional we obtain that the limit configuration u solves (3.4); the equality (3.5) follows from standard properties of relaxation. □

The remaining part of this section is devoted to the proof of Theorem 3.1. Since by Proposition 2.2 the nonlocal part $\mathcal{N}$ of the energy is continuous with respect to the convergence in $\mathcal{X}$, it is sufficient to compute the relaxation of the surface energy. This is proved, as usual, in two steps: denoting by $\mathcal{F}$ the right-hand side of (3.2), in the first step (Proposition 3.8) it is shown that the energy $\mathcal{F}(u)$ is smaller than the liminf of the energies of every sequence approximating u; in the second step (Proposition 3.9), we prove the sharpness of the lower bound, constructing a recovery sequence made of regular configurations.

3.1 Lower semicontinuity

The lower semicontinuity of the interface part of the energy (3.2) follows essentially from the same type of arguments as in [3]. It is indeed well-known (see also [39]) that, for an isotropic surface energy defined on Caccioppoli partitions of a domain Ω, where each interface has a cost proportional to its area, the validity of the triangle inequalities between the surface tensions is a necessary and sufficient condition for the lower semicontinuity of the functional. However, in our case we do not deal with generic Caccioppoli partitions, but we have a geometric restriction on the admissible configurations; this is reflected in the fact that the surface tension coefficients $\overline{\Psi}(i,j)$ do not satisfy all the possible triangle inequalities, but only those corresponding to actual configurations of the system. Hence we cannot directly deduce the following lower semicontinuity result from [3], and we prefer to give a self-contained proof, based on the same type of argument.

For the proof we will need the following lower semicontinuity lemma.

Lemma 3.6. *Let $F^1, F^2 \subset B_1$ be disjoint sets of finite perimeter with $F^1 \cup F^2 = B_1$, and let $m > 2$. Suppose that, for $i, j \in \{1, \dots, m\}$, $\lambda_{ij} = \lambda_{ji}$ are nonnegative coefficients such that*

$$\lambda_{12} \le \lambda_{1i_1} + \lambda_{i_1 i_2} + \ldots + \lambda_{i_{j-1} i_j} + \lambda_{i_j 2} \qquad \text{for all } i_1, \dots i_j \in \{3, \dots, m\} \text{ distinct.} \tag{3.6}$$

For every $k \in \mathbb{N}$ let $(F_k^1, F_k^2, \dots, F_k^m)$ be a Caccioppoli partition of B_1 into m sets, such that $F_k^1 \to F^1$, $F_k^2 \to F^2$, and $F_k^i \to \emptyset$ in $L^1(B_1)$, $i = 3, \dots, m$, as $k \to \infty$. Then

$$\lambda_{12}\mathcal{H}^{n-1}(\partial^* F^1 \cap \partial^* F^2) \le \liminf_{k\to\infty} \sum_{\substack{i,j=1\\ i<j}}^m \lambda_{ij}\mathcal{H}^{n-1}(\partial^* F_k^i \cap \partial^* F_k^j)\,.$$

Proof. The idea is that the triangle inequalities (3.6) imply the following property, introduced by Morgan [28]: given any Caccioppoli partition $(E^1, \dots, E^m)$ of B_1, there exists $I \subset \{3, \dots, m\}$ such that

$$\lambda_{12}\mathscr{P}\Big(E^1 \cup \bigcup_{i\in I} E^i; B_1\Big) \le \sum_{\substack{i,j=1\\ i<j}}^m \lambda_{ij}\mathcal{H}^{n-1}(\partial^* E^i \cap \partial^* E^j)\,. \tag{3.7}$$

In other words, interpreting the sets of the partition as immiscible fluids, it is possible to take away all but the first two fluids, and to fill the region left empty by the other fluids, reshaping to a configuration with less energy. Such property is well-known assuming that *all* the possible triangle inequalities among the coefficients are satisfied; an inspection of the proof of [28, Proposition 3.1] shows that, if we want to keep only the first two fluids, then the triangle inequalities (3.6) are sufficient.

From this property, the statement of the lemma follows easily by lower semicontinuity of the perimeter. □

Remark 3.7. *In the case $m = 3$ it is possible to give a short direct proof of the above result without the use of graph theory, as in [28]. Indeed, if we denote by $L_k^{ij} := \mathcal{H}^{n-1}(\partial^* F_k^i \cap \partial^* F_k^j)$, then*

$$\begin{aligned}\sum_{\substack{i,j=1,2,3\\ i\neq j}} \lambda_{ij} L_k^{ij} &\ge \lambda_{12} L_k^{12} + \big(\lambda_{13} + \lambda_{23}\big)\min\{L_k^{13}, L_k^{23}\}\\ &\ge \lambda_{12}\min\big\{L_k^{12} + L_k^{13}, L_k^{12} + L_k^{23}\big\}\\ &= \lambda_{12}\min\big\{\mathscr{P}(F_k^1; B_1), \mathscr{P}(F_k^2; B_1)\big\}\,.\end{aligned}$$

Thus, using the identity $\mathscr{P}(F^1; B_1) = \mathscr{P}(F^2; B_1) = \mathcal{H}^{n-1}(\partial^* F^1 \cap \partial^* F^2)$, the convergences $F_k^1 \to F^1$, $F_k^2 \to F^2$, and the lower semicontinuity of the perimeter, we get the result.

Proposition 3.8. *Denote by $\mathcal{F}$ the right-hand side of (3.2). For every $u \in \mathcal{X}$ and for every sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that $u_j \to u$ in $\mathcal{X}$ there holds*

$$\mathcal{F}(u) \le \liminf_{j\to\infty} \mathscr{F}(u_j). \tag{3.8}$$

Proof. As already observed, it is sufficient to consider the surface part of the energy, as the nonlocal term is continuous with respect to the convergence in $\mathcal{X}$. Without loss of generality we can assume that the sequence $\mathscr{F}(u_j)$ is bounded and that the measures $\mu_j := \Psi(u_j^+, u_j^-)\mathcal{H}^{n-1} \llcorner J_{u_j}$ locally weakly* converge in $\mathbb{R}^n$ to a positive Radon measure μ. We need to show that $\mu \geq \overline{\Psi}(u^+, u^-)\mathcal{H}^{n-1} \llcorner J_u$. By [4, Theorem 2.56] it is sufficient to show that

$$\limsup_{\rho\to 0^+} \frac{\mu(B_\rho(x))}{\omega_{n-1}\rho^{n-1}} \geq \overline{\Psi}(u^+(x), u^-(x)) \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u. \tag{3.9}$$

This can be proved by a blow-up argument: we fix $x \in J_u$ and we let $\rho_k \to 0^+$ be a sequence such that $\mu(\partial B_{\rho_k}(x)) = 0$, and we select a subsequence $(u_{j_k})_k$ such that

$$\frac{1}{\rho_k^n}\int_{B_{\rho_k}(x)} |u_{j_k} - u| \,\mathrm{d}x \leq \rho_k, \qquad \mu_{j_k}(B_{\rho_k}(x)) \leq \mu(\bar{B}_{\rho_k}(x)) + \rho_k^n.$$

We introduce the rescaled functions $v_k(y) := u_{j_k}(x + \rho_k y)$, $w^k(y) := u(x + \rho_k y)$, defined in B_1; then $w^k \to w$ in $L^1(B_1)$ as $k \to \infty$, where

$$w(y) := \begin{cases} u^+(x) & \text{in } \{y \in B_1 : y \cdot \nu_u(x) > 0\}, \\ u^-(x) & \text{in } \{y \in B_1 : y \cdot \nu_u(x) < 0\}. \end{cases}$$

Moreover by our choice of the subsequence

$$\int_{B_1} |v_k - w| \,\mathrm{d}y \leq \int_{B_1} |v_k - w^k| \,\mathrm{d}y + \int_{B_1} |w^k - w| \,\mathrm{d}y \leq \rho_k + \int_{B_1} |w^k - w| \,\mathrm{d}y,$$

that is, also $v_k \to w$ in $L^1(B_1)$ as $k \to \infty$. Noticing that

$$\begin{aligned} \limsup_{\rho\to 0^+} \frac{\mu(B_\rho(x))}{\omega_{n-1}\rho^{n-1}} &\geq \liminf_{k\to\infty} \frac{\mu(\bar{B}_{\rho_k}(x))}{\omega_{n-1}\rho_k^{n-1}} \geq \liminf_{k\to\infty} \frac{\mu_{j_k}(B_{\rho_k}(x))}{\omega_{n-1}\rho_k^{n-1}} \\ &= \liminf_{k\to\infty} \frac{1}{\omega_{n-1}\rho_k^{n-1}} \int_{B_{\rho_k}(x)\cap J_{u_{j_k}}} \Psi(u_{j_k}^+, u_{j_k}^-) \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \liminf_{k\to\infty} \frac{1}{\omega_{n-1}} \int_{B_1 \cap J_{v_k}} \Psi(v_k^+, v_k^-) \,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

the claim (3.9) will follow once we prove that

$$\liminf_{k\to\infty} \int_{B_1\cap J_{v_k}} \Psi(v_k^+, v_k^-) \,\mathrm{d}\mathcal{H}^{n-1} \geq \omega_{n-1}\overline{\Psi}(w^+, w^-). \tag{3.10}$$

In view of (2.15), in order to show (3.9) we now have to distinguish among six possible cases, depending on which interface contains the point x.

Case 1: $x \in \Gamma_u^A$. In this case, in the blow-up limit we have that half of the ball B_1 is filled with the pure phase A_u, and the other half ball is filled with the phase V_u; that is, up to a permutation $w^+ = 1$, $w^- = 0$. Notice that $x_n > 0$ and therefore for k large enough the ball $B_{\rho_k}(x)$ does not intersect the substrate and can contain only the phases A, B, and V; hence, the rescaled functions v_k can only take the values $\{\pm 1, 0\}$ in B_1, that is, $v_k^\pm \in \{\pm 1, 0\}$. Since by definition of $\overline{\Psi}$ the triangle inequality $\overline{\Psi}(1,0) \leq \overline{\Psi}(1,-1) + \overline{\Psi}(-1,0)$ holds and $\overline{\Psi} \leq \Psi$, the claim (3.10) follows from Lemma 3.6, applied to $F^1 := \{w = 1\}$, $F^2 := \{w = 0\}$, $F_k^1 := \{v_k = 1\} \to F^1$, $F_k^2 := \{v_k = 0\} \to F^2$, $F_k^3 := \{v_k = -1\} \to \emptyset$.

Case 2: $x \in \Gamma_u^B$. This is analogous to the previous case.

Case 3: $x \in \Gamma_u^{AB}$. In this case $w^+ = 1$, $w^- = -1$, and since $B_{\rho_k}(x)$ does not intersect the substrate for k large enough, we have $v_k^\pm \in \{\pm 1, 0\}$. Then (3.10) follows again by Lemma 3.6 in view of the triangle inequality $\overline{\Psi}(1,-1) \le \overline{\Psi}(1,0) + \overline{\Psi}(-1,0)$, which holds by definition of $\overline{\Psi}$ and by the assumption $\sigma_{AB} \le \sigma_A + \sigma_B$.

Case 4: $x \in S_u^A$. In this case $w^+ = 1$, $w^- = 2$. In principle, all the four phases can be present in a neighbourhood of the point x; however, by the geometric constraint the limit interface between the phase A and the substrate S cannot be approximated by the boundary of V. Therefore, in order to apply Lemma 3.6, we first need to get rid of the possible infiltration of the phase V.

We denote by $A_k := \{v_k = 1\}$, $B_k := \{v_k = -1\}$, $V_k := \{v_k = 0\}$ the phases of v_k in the upper half ball B_1^+, and the corresponding interfaces by

$$\Gamma_k^A := \partial^* A_k \cap \partial^* V_k \cap B_1\,, \quad \Gamma_k^B := \partial^* B_k \cap \partial^* V_k \cap B_1\,, \quad \Gamma_k^{AB} := \partial^* A_k \cap \partial^* B_k \cap B_1,$$

$$S_k^A := \partial^* A_k \cap \partial S \cap B_1, \quad S_k^B := \partial^* B_k \cap S \cap B_1, \quad S_k^V := \partial^* V_k \cap S \cap B_1.$$

Then we modify v_k by "filling" the region V_k with either A_k or B_k, according to the following rule:

$$\tilde{v}_k(y) := \begin{cases} v_k(y) & \text{if } y \in B_1 \setminus V_k, \\ 1 & \text{if } y \in V_k \text{ and } \mathcal{H}^{n-1}(\Gamma_k^B) \le \mathcal{H}^{n-1}(\Gamma_k^A), \\ -1 & \text{if } y \in V_k \text{ and } \mathcal{H}^{n-1}(\Gamma_k^A) < \mathcal{H}^{n-1}(\Gamma_k^B). \end{cases}$$

Notice that $\tilde{v}_k \to w$ in $L^1(B_1)$, and that the partition of the unit ball determined by $\tilde{v}_k$ does not contain the phase V. Therefore, using the inequality $\Psi(1,0) + \Psi(-1,0) \ge \Psi(-1,1)$,

$$\begin{aligned}
\int_{B_1 \cap J_{v_k}} \Psi(v_k^+, v_k^-)\, \mathrm{d}\mathcal{H}^{n-1} &= \Psi(1,0)\mathcal{H}^{n-1}(\Gamma_k^A) + \Psi(-1,0)\mathcal{H}^{n-1}(\Gamma_k^B) + \Psi(-1,1)\mathcal{H}^{n-1}(\Gamma_k^{AB}) \\
&\quad + \Psi(1,2)\mathcal{H}^{n-1}(S_k^A) + \Psi(-1,2)\mathcal{H}^{n-1}(S_k^B) + \Psi(0,2)\mathcal{H}^{n-1}(S_k^V) \\
&\ge \Psi(-1,1)\min\{\mathcal{H}^{n-1}(\Gamma_k^A), \mathcal{H}^{n-1}(\Gamma_k^B)\} + \Psi(-1,1)\mathcal{H}^{n-1}(\Gamma_k^{AB}) \\
&\quad + \Psi(1,2)\mathcal{H}^{n-1}(S_k^A) + \Psi(-1,2)\mathcal{H}^{n-1}(S_k^B) + \Psi(0,2)\mathcal{H}^{n-1}(S_k^V) \\
&= \int_{B_1 \cap J_{\tilde{v}_k}} \Psi(\tilde{v}_k^+, \tilde{v}_k^-)\, \mathrm{d}\mathcal{H}^{n-1} + \Psi(0,2)\mathcal{H}^{n-1}(S_k^V) \\
&\quad - \max\{\Psi(1,2), \Psi(-1,2)\}\mathcal{H}^{n-1}(S_k^V).
\end{aligned}$$

By observing that $\mathcal{H}^{n-1}(S_k^V) \to 0$ as $k \to \infty$, from the previous inequality we obtain

$$\liminf_{k\to\infty} \int_{B_1 \cap J_{v_k}} \Psi(v_k^+, v_k^-)\, \mathrm{d}\mathcal{H}^{n-1} \ge \liminf_{k\to\infty} \int_{B_1 \cap J_{\tilde{v}_k}} \Psi(\tilde{v}_k^+, \tilde{v}_k^-)\, \mathrm{d}\mathcal{H}^{n-1}.$$

To deduce (3.10) we can now apply Lemma 3.6 to the partition of B_1 determined by $\tilde{v}_k$, which contains only the three phases A, B, S and that converges to the configuration where the upper half-ball is filled by A, and the lower half-ball is filled by S. Therefore to apply Lemma 3.6 one only needs to check the triangle inequality $\overline{\Psi}(1,2) \le \overline{\Psi}(1,-1) + \overline{\Psi}(-1,2)$, which holds by definition of $\overline{\Psi}$.

Case 5: $x \in S_u^B$. This is analogous to Case 4, with the roles of phases A and B exchanged.

Case 6: $x \in S_u^V$. In this case $w^+ = 0$, $w^- = 2$, and all the four phases can be present in a neighbourhood of the point x. We deduce (3.10) by applying once more Lemma 3.6: one only needs to check that all the possible triangle inequalities (3.6) hold for $\lambda_{12} = \overline{\Psi}(0,2)$, namely

$$\overline{\Psi}(0,2) \le \overline{\Psi}(0,1) + \overline{\Psi}(1,2), \qquad \overline{\Psi}(0,2) \le \overline{\Psi}(0,-1) + \overline{\Psi}(-1,2),$$

$$\overline{\Psi}(0,2) \le \overline{\Psi}(0,1) + \overline{\Psi}(1,-1) + \overline{\Psi}(-1,2), \qquad \overline{\Psi}(0,2) \le \overline{\Psi}(0,-1) + \overline{\Psi}(-1,1) + \overline{\Psi}(1,2).$$

All these inequalities can be checked by using the definition of $\overline{\Psi}$. □

3.2 Recovery sequence

The goal of this section is to prove the following result, which combined with Proposition 3.8 completes the proof of Theorem 3.1.

Proposition 3.9. *Denote by $\mathcal{F}$ the right-hand side of (3.2). For every $u \in \mathcal{X}$ there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that $u_j \to u$ in $\mathcal{X}$, $|A_{u_j}| = |A_u|$, $|B_{u_j}| = |B_u|$, and*

$$\mathcal{F}(u) = \lim_{j\to\infty} \mathscr{F}(u_j). \tag{3.11}$$

The proof we present uses an approximation result proved in [9] (the proof of the first two statements is contained in Step 1 of the proof of [9, Proposition 4.1], while the last statement is proved in [9, Remark 4.4]).

Lemma 3.10. *Let $h \in \mathcal{AP}(Q_L)$. Then, for every $\varepsilon > 0$, there exists $f \in C^\infty(\mathbb{R}^{n-1})$, Q_L-periodic, such that*

$$\|f - h\|_{L^1(Q_L)} + \mathcal{H}^{n-1}(\Gamma_h \cap \Omega_f) \le \varepsilon,$$

$$\left| \int_{Q_L} \sqrt{1 + |\nabla f(x')|^2}\, \mathrm{d}x' - \mathcal{H}^{n-1}(\Gamma_h) \right| \le \varepsilon,$$

and

$$\left| \mathcal{H}^{n-1}(\{f = 0\}) - \mathcal{H}^{n-1}(\{h = 0\}) \right| \le \varepsilon.$$

We are now in position to prove the existence of a recovery sequence.

Proof of Proposition 3.9. Fix $u \in \mathcal{X}$ and let $h_u \in \mathcal{AP}(Q_L)$ be the corresponding admissible profile. The proof is divided into several steps (see Figure 2 for the modifications performed in Step 2, 3, and 4).

Step 1: approximation of h_u with a regular profile. In this step we construct a sequence $\widetilde{u}_j \in \mathcal{X}$ such that $\widetilde{u}_j \to u$ in $\mathcal{X}$ and $\mathcal{F}(\widetilde{u}_j) \to \mathcal{F}(u)$, with the additional property that the corresponding profiles $h_{\widetilde{u}_j}$ are smooth. By a diagonal argument this will allow us, in the following steps, to work under the assumption that the limiting profile is smooth, and to construct a recovery sequence only in this case.

For each $j \in \mathbb{N}$ let $f_j \in C^\infty(\mathbb{R}^{n-1})$ be the Q_L-periodic function given by Lemma 3.10 relative to $\varepsilon := \frac{1}{j}$. In particular, we have

$$\|f_j - h_u\|_{L^1(Q_L)} + \mathcal{H}^{n-1}(\Gamma_{h_u} \cap \Omega_{f_j}) \le \frac{1}{j}\,, \tag{3.12}$$

$$\left| \int_{Q_L} \sqrt{1 + |\nabla f_j(x')|^2}\, \mathrm{d}x' - \mathcal{H}^{n-1}(\Gamma_{h_u}) \right| \leq \frac{1}{j}\,, \tag{3.13}$$

and

$$\left| \, \mathcal{H}^{n-1}(\{f_j = 0\}) - \mathcal{H}^{n-1}(\{h_u = 0\}) \, \right| \leq \frac{1}{j}\,. \tag{3.14}$$

Define the function $\widetilde{u}_j : \mathbb{R}^n \to \{0, -1, 1, 2\}$ as

$$\widetilde{u}_j(x) := \begin{cases} u(x) & \text{if } x \in \Omega^\#_{f_j} \cap \Omega^\#_{h_u}, \\ 1 & \text{if } x \in \Omega^\#_{f_j} \backslash \Omega^\#_{h_u}, \\ 0 & \text{if } x \in \mathbb{R}^n_+ \backslash \Omega^\#_{f_j}, \\ 2 & \text{if } x \in S. \end{cases} \tag{3.15}$$

This modification amounts to fill the (small) region in $\Omega_{f_j} \backslash \Omega_{h_u}$ by the phase A, and to remove the possible parts of the phases A and B outside Ω_{f_j} by replacing them with V. Notice that $\widetilde{u}_j \in \mathcal{X}_{\mathrm{reg}}$ with $f_j = h_{\widetilde{u}_j}$ and

$$\|\widetilde{u}_j - u\|_{L^1(Q_L \times \mathbb{R})} \leq \frac{2}{j}\,. \tag{3.16}$$

First, we show that

$$\mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(\Gamma^{AB}_u). \tag{3.17}$$

Define the Radon measures $\mu^A := D\chi_{A^\#_u}$, $\mu^B := D\chi_{B^\#_u}$, and, for $j \in \mathbb{N}$, define $\mu^A_j := D\chi_{A^\#_{\widetilde{u}_j}}$ and $\mu^B_j := D\chi_{B^\#_{\widetilde{u}_j}}$. Then (3.13) and (3.14) yield

$$\lim_{j\to\infty} |\mu^A_j + \mu^B_j|(Q^+_L) = \lim_{j\to\infty} |D\chi_{\Omega^\#_{f_j}}|(Q^+_L) = |D\chi_{\Omega^\#_{h_u}}|(Q^+_L) = |\mu^A + \mu^B|(Q^+_L), \tag{3.18}$$

and, since $A_{\widetilde{u}_j} \to A_u$, $B_{\widetilde{u}_j} \to B_u$, using also the periodicity,

$$|\mu^A|(Q^+_L) \leq \liminf_{j\to\infty} |\mu^A_j|(Q^+_L), \qquad |\mu^B|(Q^+_L) \leq \liminf_{j\to\infty} |\mu^B_j|(Q^+_L). \tag{3.19}$$

Combining the previous estimates we get

$$\begin{aligned} \mathcal{H}^{n-1}(\Gamma^{AB}_u) &= \frac{|\mu^A|(Q^+_L) + |\mu^B|(Q^+_L) - |\mu^A + \mu^B|(Q^+_L)}{2} \\ &\leq \liminf_{j\to\infty} \frac{|\mu^A_j|(Q^+_L) + |\mu^B_j|(Q^+_L) - |\mu^A_j + \mu^B_j|(Q^+_L)}{2} \\ &= \liminf_{j\to\infty} \mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j}). \end{aligned}$$

On the other hand, by construction and by (3.12)

$$\mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j}) \leq \mathcal{H}^{n-1}(\Gamma^{AB}_u) + \mathcal{H}^{n-1}(\Gamma_{h_u} \cap \Omega_{f_j}) \leq \mathcal{H}^{n-1}(\Gamma^{AB}_u) + \frac{1}{j}\,,$$

which together with the previous estimate proves (3.17).

Next, we claim that

$$\mathcal{H}^{n-1}(\Gamma^A_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(\Gamma^A_u), \qquad \mathcal{H}^{n-1}(\Gamma^B_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(\Gamma^B_u), \tag{3.20}$$

and that

$$\mathcal{H}^{n-1}(S^A_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(S^A_u), \qquad \mathcal{H}^{n-1}(S^B_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(S^B_u). \tag{3.21}$$

Using (3.17), (3.18), and (3.19), we have

$$\begin{aligned}
|\mu^A|(Q_L^+) + |\mu^B|(Q_L^+) &\le \liminf_{j\to\infty} \big[\, |\mu_j^A|(Q_L^+) + |\mu_j^B|(Q_L^+)\,\big] \\
&= \liminf_{j\to\infty} \big[\, |\mu_j^A + \mu_j^B|(Q_L^+) + 2\mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j})\,\big] \\
&= |\mu^A + \mu^B|(Q_L^+) + 2\mathcal{H}^{n-1}(\Gamma^{AB}_u) \\
&= |\mu^A|(Q_L^+) + |\mu^B|(Q_L^+),
\end{aligned}$$

hence

$$|\mu_j^A|(Q_L^+) \to |\mu^A|(Q_L^+), \qquad |\mu_j^B|(Q_L^+) \to |\mu^B|(Q_L^+). \tag{3.22}$$

Denote now, for $\varepsilon > 0$, $Q^\varepsilon := Q_L \times (\varepsilon, +\infty)$, and notice that for $\mathscr{L}^1$-almost every $\varepsilon > 0$ we have $\mathcal{H}^{n-1}(J_u \cap \{x_n = \varepsilon\}) = 0$. For all such ε, thanks to (3.18) and to (3.22) we obtain

$$|\mu_j^A|(Q^\varepsilon) \to |\mu^A|(Q^\varepsilon), \qquad |\mu_j^B|(Q^\varepsilon) \to |\mu^B|(Q^\varepsilon), \qquad |\mu_j^A + \mu_j^B|(Q^\varepsilon) \to |\mu^A + \mu^B|(Q^\varepsilon),$$

and in turn, arguing as in the proof of (3.17),

$$\mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j} \cap Q^\varepsilon) \to \mathcal{H}^{n-1}(\Gamma^{AB}_u \cap Q^\varepsilon).$$

Then for almost every $\varepsilon > 0$

$$\begin{aligned}
\mathcal{H}^{n-1}(\Gamma^A_{\widetilde{u}_j} \cap Q^\varepsilon) &= |\mu_j^A|(Q^\varepsilon) - \mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j} \cap Q^\varepsilon) \\
&\to |\mu^A|(Q^\varepsilon) - \mathcal{H}^{n-1}(\Gamma^{AB}_u \cap Q^\varepsilon) = \mathcal{H}^{n-1}(\Gamma^A_u \cap Q^\varepsilon),
\end{aligned}$$

and similarly $\mathcal{H}^{n-1}(\Gamma^B_{\widetilde{u}_j} \cap Q^\varepsilon) \to \mathcal{H}^{n-1}(\Gamma^B_u \cap Q^\varepsilon)$. From these two convergences (3.20) follows: indeed, if (3.20) fails then for some $\eta > 0$ we would have (using the fact that $\mathcal{H}^{n-1}(\Gamma^A_{\widetilde{u}_j} \cup \Gamma^B_{\widetilde{u}_j}) \to \mathcal{H}^{n-1}(\Gamma^A_u \cup \Gamma^B_u)$)

$$\limsup_{j\to\infty} \mathcal{H}^{n-1}(\Gamma^A_{\widetilde{u}_j}) \ge \mathcal{H}^{n-1}(\Gamma^A_u) + \eta, \qquad \liminf_{j\to\infty} \mathcal{H}^{n-1}(\Gamma^B_{\widetilde{u}_j}) \le \mathcal{H}^{n-1}(\Gamma^B_u) - \eta$$

(or the symmetric inequalities with A and B exchanged). This yields

$$\liminf_{j\to\infty} \mathcal{H}^{n-1}(\Gamma^B_{\widetilde{u}_j} \setminus Q^\varepsilon) \le \mathcal{H}^{n-1}(\Gamma^B_u \setminus Q^\varepsilon) - \eta \qquad \text{for every } \varepsilon > 0,$$

which is a contradiction since $\mathcal{H}^{n-1}(\Gamma^B_u \setminus Q^\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves (3.20).

Finally, by writing

$$\begin{aligned}
|\mu_j^A|(Q_L^+) &= \mathcal{H}^{n-1}(\Gamma^A_{\widetilde{u}_j}) + \mathcal{H}^{n-1}(\Gamma^{AB}_{\widetilde{u}_j}) + \mathcal{H}^{n-1}(S^A_{\widetilde{u}_j}), \\
|\mu^A|(Q_L^+) &= \mathcal{H}^{n-1}(\Gamma^A_u) + \mathcal{H}^{n-1}(\Gamma^{AB}_u) + \mathcal{H}^{n-1}(S^A_u)
\end{aligned}$$

(and similarly for B), we conclude that also (3.21) holds by using (3.17), (3.20), and (3.22).

Thanks to (3.16), (3.17), (3.20), and (3.21) we obtain $\mathcal{F}(\widetilde{u}_j) \to \mathcal{F}(u)$, as desired.

Step 2: the non-exposed substrate. Assume $v \in \mathcal{X}_{\mathrm{reg}}$. We construct a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that

$$\lim_{j\to\infty} \|v_j - v\|_{L^1(Q_L^+)} = 0, \tag{3.23}$$

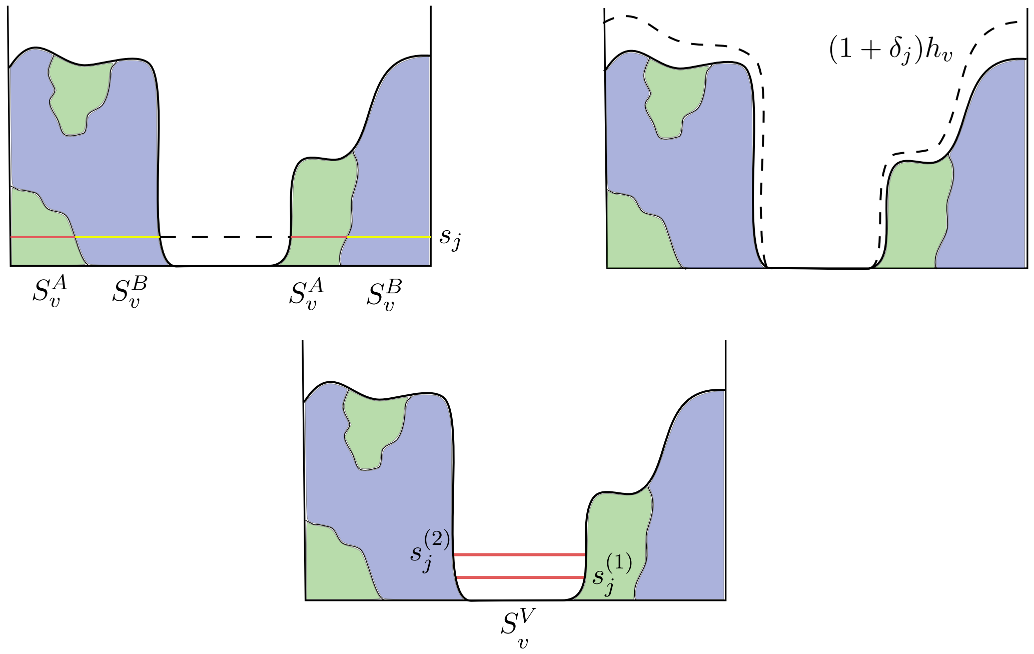

Figure 2: The three figures depict the modifications we perform in Step 2, 3, and 4 respectively. *Top-left* (Step 2): we consider levels $\{s_j\}_{j\in\mathbb{N}}$ so that $\mathcal{H}^{n-1}(A_v \cap \{x_n = s_j\})$ converges to $\mathcal{H}^{n-1}(S_v^A)$, and $\mathcal{H}^{n-1}(B_v \cap \{x_n = s_j\})$ converges to $\mathcal{H}^{n-1}(S_v^B)$. Then the part of the subgraph of h_v that is below the threshold s_j is filled with the phase A. *Top-right* (Step 3): we rescale by $1+\delta_j$ the part of the graph that does not touch the substrate. *Bottom* (Step 4): we consider two sequences of levels $s_j^{(1)} < s_j^{(2)}$ so that $\mathcal{H}^{n-1}(V_v \cap \{x_n = s_j^{(i)}\})$ converges to $\mathcal{H}^{n-1}(S_v^V)$. The part of V below the threshold $s_j^{(1)}$ is filled with the phase A, while the part of V between the two levels is filled with B.

that allows to recover the relaxed coefficients $\overline{\Psi}(1,2)$ and $\overline{\Psi}(-1,2)$ with the non-exposed substrate in the limit energy, in the sense that

$$\mathscr{F}(v_j) \to \mathscr{F}(v) + \big(\overline{\Psi}(1,2) - \Psi(1,2)\big)\mathcal{H}^{n-1}(S_v^A) + \big(\overline{\Psi}(-1,2) - \Psi(-1,2)\big)\mathcal{H}^{n-1}(S_v^B). \tag{3.24}$$

In the case where

$$\sigma_{AS} \le \sigma_{BS} + \sigma_{AB} \qquad \text{and} \qquad \sigma_{BS} \le \sigma_{AS} + \sigma_{AB}\,,$$

the relaxed surface tensions $\overline{\Psi}(1,2)$ and $\overline{\Psi}(-1,2)$ coincide with the original ones $\Psi(1,2)$ and $\Psi(-1,2)$; in this case there is nothing to do, and we just take $v_j := v$ for each $j \in \mathbb{N}$. Assume instead

$$\sigma_{AS} \le \sigma_{BS} + \sigma_{AB} \qquad \text{and} \qquad \sigma_{AS} + \sigma_{AB} < \sigma_{BS}\,.$$

The only other possible case is $\sigma_{BS} + \sigma_{AB} < \sigma_{AS}$ and $\sigma_{BS} \le \sigma_{AS} + \sigma_{AB}$, that can be treated similarly. We need to build a sequence $\{v_j\}_{j\in\mathbb{N}}$ satisfying (3.23) and (3.24), which in this case becomes

$$\mathscr{F}(v_j) \to \mathscr{F}(v) + (\sigma_{AS} + \sigma_{AB} - \sigma_{BS})\mathcal{H}^{n-1}(S_v^B). \tag{3.25}$$

By standard results on traces of BV-functions (for instance, combining equation (2.8) and Theorem 2.11 in [16]), it is possible to find a sequence $\{s_j\}_{j\in\mathbb{N}}$ with $s_j \to 0^+$ as $j \to \infty$ such that

$$\mathcal{H}^{n-1}(A_v^{(1)} \cap \{x_n = s_j\}) \to \mathcal{H}^{n-1}(S_v^A), \quad \mathcal{H}^{n-1}(B_v^{(1)} \cap \{x_n = s_j\}) \to \mathcal{H}^{n-1}(S_v^B), \tag{3.26}$$

and since $\mathcal{H}^{n-1}(J_v \cap \{x_n = t\}) = 0$ for $\mathscr{L}^1$-a.e. t we can also assume

$$\mathcal{H}^{n-1}(J_v \cap \{x_n = s_j\}) = 0. \tag{3.27}$$

Also note that, since $\mathcal{H}^{n-1}(J_v) < \infty$,

$$\mathcal{H}^{n-1}(J_v \cap \{0 < x_n < s_j\}) \to 0. \tag{3.28}$$

Define the function $v_j : \mathbb{R}^n \to \{0, -1, 1, 2\}$ as

$$v_j(x', x_n) := \begin{cases} 1 & \text{if } (x', x_n) \in \Omega_{h_v}^{\#} \text{ and } 0 < x_n < s_j, \\ v(x', x_n) & \text{otherwise,} \end{cases} \tag{3.29}$$

which satisfies $v_j \in \mathcal{X}_{\mathrm{reg}}$ for each $j \in \mathbb{N}$ (since $h_{v_j} = h_v$) and

$$\|v_j - v\|_{L^1(Q_L^+)} \le 2 s_j \mathscr{L}^{n-1}(Q_L),$$

which gives (3.23). This sequence allows to adjust the surface tensions for the substrate: namely, we have by (3.27)

$$\begin{aligned}\mathscr{F}(v_j) - \mathscr{F}(v) = (\sigma_A - \sigma_B)&\mathcal{H}^{n-1}(\Gamma_v^B \cap \{0 < x_n < s_j\}) - \sigma_{AB}\mathcal{H}^{n-1}(\Gamma_v^{AB} \cap \{0 < x_n < s_j\}) \\ &+ \sigma_{AB}\mathcal{H}^{n-1}(B_v^{(1)} \cap \{x_n = s_j\}) + (\sigma_{AS} - \sigma_{BS})\mathcal{H}^{n-1}(S_v^B) \\ &+ \gamma\,(\mathcal{N}(v_j) - \mathcal{N}(v)).\end{aligned}$$

By passing to the limit as $j \to \infty$, the first two terms on the right-hand side vanish thanks to (3.28), the third term tends to $\sigma_{AB}\mathcal{H}^{n-1}(S_v^B)$ by (3.26), and the last term tends to zero by (3.23). Hence (3.25) follows.

Step 3: the graph. Let $v \in \mathcal{X}_{\mathrm{reg}}$ and $\{v_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ be the sequence constructed in the previous step, satisfying (3.23) and (3.24). We want to modify the sequence in such a way to recover the relaxed surface tensions $\overline{\Psi}(1,0)$ and $\overline{\Psi}(-1,0)$ between the two phases A, B and the void V: more precisely, we want to construct another sequence $\{w_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that

$$\lim_{j\to\infty} \|w_j - v_j\|_{L^1(Q_L^+)} = 0, \tag{3.30}$$

and

$$\begin{aligned}\lim_{j\to\infty} \big|\mathscr{F}(w_j) - \mathscr{F}(v_j) - \big(\overline{\Psi}(1,0) - \Psi(1,0)\big)&\mathcal{H}^{n-1}(\Gamma_v^A) \\ &- \big(\overline{\Psi}(-1,0) - \Psi(-1,0)\big)\mathcal{H}^{n-1}(\Gamma_v^B)\big| = 0.\end{aligned} \tag{3.31}$$

In the case where

$$\sigma_A \le \sigma_B + \sigma_{AB}\,, \qquad \sigma_B \le \sigma_A + \sigma_{AB}$$

the relaxed surface tensions $\overline{\Psi}(1,0)$ and $\overline{\Psi}(-1,0)$ coincide with the original ones $\Psi(1,0)$ and $\Psi(-1,0)$; in this case there is nothing to do, and we just take $w_j := v_j$ for each $j \in \mathbb{N}$. Assume instead

$$\sigma_A \leq \sigma_B + \sigma_{AB}\,, \qquad \sigma_A + \sigma_{AB} < \sigma_B\,.$$

The only other possible case is $\sigma_B + \sigma_{AB} < \sigma_A$ and $\sigma_B \leq \sigma_A + \sigma_{AB}$, that can be treated similarly. In this case the condition (3.31) becomes

$$\lim_{j\to\infty} \big|\mathscr{F}(w_j) - \mathscr{F}(v_j) - (\sigma_A + \sigma_{AB} - \sigma_B)\mathcal{H}^{n-1}(\Gamma_v^B)\big| = 0. \tag{3.32}$$

Let $\delta_j \to 0^+$ and define, for each $j \in \mathbb{N}$, the function $w_j : \mathbb{R}^n \to \{0,-1,1,2\}$ by

$$w_j(x',x_n) := \begin{cases} v_j(x',x_n) & \text{if } (x',x_n) \in \Omega^\#_{h_{v_j}}, \\ 1 & \text{if } h_{v_j}(x') < x_n < (1+\delta_j)h_{v_j}(x'), \\ 0 & \text{if } x_n \geq (1+\delta_j)h_{v_j}(x'). \end{cases} \tag{3.33}$$

Note that $h_{w_j} = (1+\delta_j)h_{v_j} = (1+\delta_j)h_v$ (recalling that $h_{v_j} = h_v$ for all j, by the construction in Step 2), therefore $w_j \in \mathcal{X}_{\mathrm{reg}}$ and

$$\|w_j - v_j\|_{L^1(Q_L^+)} \leq \int_{Q_L}\int_{h_v(x')}^{(1+\delta_j)h_v(x')} |1 - v_j(x',x_n)|\,\mathrm{d}x_n\,\mathrm{d}x' \leq 2\delta_j \int_{Q_L} h_v(x')\,\mathrm{d}x'$$

which yields (3.30). Moreover by a Taylor expansion

$$\begin{aligned}\int_{Q_L\setminus S_v^V} \sqrt{1+|(1+\delta_j)\nabla h_v(x')|^2}\,\mathrm{d}x' &= \int_{Q_L\setminus S_v^V} \sqrt{1+|\nabla h_v(x')|^2}\,\mathrm{d}x' + o(1) \\ &= \mathcal{H}^{n-1}(\Gamma_{v_j}^A) + \mathcal{H}^{n-1}(\Gamma_{v_j}^B) + o(1),\end{aligned}$$

therefore

$$\begin{aligned}\mathscr{F}(w_j) - \mathscr{F}(v_j) &= \sigma_A \int_{Q_L\setminus S_v^V} \sqrt{1+|(1+\delta_j)\nabla h_v(x')|^2}\,\mathrm{d}x' \\ &\qquad + (\sigma_{AB} - \sigma_B)\mathcal{H}^{n-1}(\Gamma_{v_j}^B) - \sigma_A\mathcal{H}^{n-1}(\Gamma_{v_j}^A) + \gamma\,(\mathcal{N}(w_j) - \mathcal{N}(v_j)) \\ &= \sigma_A\big(\mathcal{H}^{n-1}(\Gamma_{v_j}^A) + \mathcal{H}^{n-1}(\Gamma_{v_j}^B)\big) + o(1) \\ &\qquad + (\sigma_{AB} - \sigma_B)\mathcal{H}^{n-1}(\Gamma_{v_j}^B) - \sigma_A\mathcal{H}^{n-1}(\Gamma_{v_j}^A) + \gamma\,(\mathcal{N}(w_j) - \mathcal{N}(v_j)).\end{aligned}$$

We get (3.32) by using (3.30) and recalling that, by the construction in Step 2, we have $\mathcal{H}^{n-1}(\Gamma_{v_j}^A) \to \mathcal{H}^{n-1}(\Gamma_v^A)$ and $\mathcal{H}^{n-1}(\Gamma_{v_j}^B) \to \mathcal{H}^{n-1}(\Gamma_v^B)$.

Step 4: the exposed substrate. Let $v \in \mathcal{X}_{\mathrm{reg}}$ and $\{w_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ be the sequence constructed in the previous step. We want to modify again the sequence in such a way to recover the relaxed surface tension $\overline{\Psi}(0,2)$ of the exposed substrate, that is the interface between the substrate S and the void V: more precisely, we want to construct another sequence $\{z_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that

$$\lim_{j\to\infty} \|z_j - w_j\|_{L^1(Q_L^+)} = 0, \tag{3.34}$$

and

$$\lim_{j\to\infty} \big|\mathscr{F}(z_j) - \mathscr{F}(w_j) - \big(\overline{\Psi}(0,2) - \Psi(0,2)\big)\mathcal{H}^{n-1}(S_v^V)\big| = 0. \tag{3.35}$$

In the case where

$$\sigma_S \le \min\{\sigma_{AS}+\bar\sigma_A, \sigma_{BS}+\bar\sigma_B\}$$

there is nothing to do since $\overline{\Psi}(0,2)=\Psi(0,2)$, and thus we define $z_j := w_j$ for all $j\in\mathbb{N}$. Assume that

$$\sigma_{AS}+\bar\sigma_A \le \min\{\sigma_S, \sigma_{BS}+\bar\sigma_B\} \qquad \text{and} \qquad \bar\sigma_A = \sigma_{AB}+\sigma_B.$$

In this case (3.35) becomes

$$\lim_{j\to\infty} \left|\mathscr{F}(z_j)-\mathscr{F}(w_j)-(\sigma_{AS}+\sigma_{AB}+\sigma_B-\sigma_S)\mathcal{H}^{n-1}(S_v^V)\right| = 0. \tag{3.36}$$

Note that the other possible cases can be treated similarly (and even more easily).

We fix two sequences $s_j^{(1)}, s_j^{(2)} \in (0,1)$, for $j\in\mathbb{N}$, with $s_j^{(1)} < s_j^{(2)}$ and $s_j^{(1)}, s_j^{(2)} \to 0$ as $j\to\infty$, such that, by setting

$$L_s := V_{w_j}\cap\{x_n = s\},$$

we have

$$\mathcal{H}^{n-1}(L_{s_j^{(1)}}) \to \mathcal{H}^{n-1}(S_v^V), \qquad \mathcal{H}^{n-1}(L_{s_j^{(2)}}) \to \mathcal{H}^{n-1}(S_v^V), \tag{3.37}$$

and

$$\mathcal{H}^{n-1}(\Gamma_{w_j}\cap\{0<x_n<s_j^{(2)}\}) \to 0. \tag{3.38}$$

The existence of such sequences can be proved similarly to (3.26), using also the convergence $\mathcal{H}^{n-1}(S_{w_j}^V) \to \mathcal{H}^{n-1}(S_v^V)$ in view of the construction of w_j in the previous step. We define the function $z_j : Q_L\times\mathbb{R}\to\{0,-1,1,2\}$ (extended by periodicity to $\mathbb{R}^n$) by

$$z_j(x',x_n) := \begin{cases} w_j(x',x_n) & \text{if } (x',x_n)\in\Omega_{h_{w_j}}\cup S, \\ 1 & \text{if } (x',x_n)\in V_{w_j} \text{ and } 0<x_n<s_j^{(1)}, \\ -1 & \text{if } (x',x_n)\in V_{w_j} \text{ and } s_j^{(1)}<x_n<s_j^{(2)}, \\ 0 & \text{else.} \end{cases} \tag{3.39}$$

Since $h_{z_j} = \max\{h_{w_j}, s_j^{(2)}\}$ we have $z_j\in\mathcal{X}_{\mathrm{reg}}$, and also $\|w_j - z_j\|_{L^1(Q_L^+)} \le s_j^{(2)}\mathscr{L}^{n-1}(Q_L)$, which yields (3.34). Moreover

$$\begin{aligned} \mathscr{F}(z_j)-\mathscr{F}(w_j) = \sigma_B\mathcal{H}^{n-1}(L_{s_j^{(2)}}) + \sigma_{AB}\mathcal{H}^{n-1}(L_{s_j^{(1)}}) + (\sigma_{AS}-\sigma_S)\mathcal{H}^{n-1}(S_{w_j}^V) \\ + \gamma\left(\mathcal{N}(z_j)-\mathcal{N}(w_j)\right) + R_j, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} R_j := -\mathcal{H}^{n-1}(\Gamma_{w_j}^A\cap\{0<x_n<s_j^{(1)}\}) + (\sigma_{AB}-\sigma_B)\mathcal{H}^{n-1}(\Gamma_{w_j}^B\cap\{0<x_n<s_j^{(1)}\}) \\ + (\sigma_{AB}-\sigma_A)\mathcal{H}^{n-1}(\Gamma_{w_j}^A\cap\{s_j^{(1)}<x_n<s_j^{(2)}\}) - \mathcal{H}^{n-1}(\Gamma_{w_j}^B\cap\{s_j^{(1)}<x_n<s_j^{(2)}\}). \end{aligned}$$

Notice that $R_j\to 0$ thanks to (3.38). We then obtain (3.36) by passing to the limit in (3.40), using (3.34), (3.37), and the fact that $\mathcal{H}^{n-1}(S_{w_j}^V)\to\mathcal{H}^{n-1}(S_v^V)$.

Step 5: the mass constraint. By combining the constructions in the previous steps and using a diagonal argument, we have that given $u \in \mathcal{X}$, there exists a sequence $\{z_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that

$$\lim_{j\to\infty} \|z_j - u\|_{L^1(Q_L^+)} = 0, \qquad \lim_{j\to\infty} \mathscr{F}(z_j) = \mathcal{F}(u) \tag{3.41}$$

(see in particular (3.23), (3.30), (3.34) for the convergence of the functions, and (3.24), (3.31), (3.35) for the convergence of the energies). In order to obtain the recovery sequence, we need to restore the mass constraint: denoting by $|\Omega_{h_u}| = M$, $|A_u| = m$, we modify the sequence $\{z_j\}_{z\in\mathbb{N}}$ and we construct a new sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{X}_{\mathrm{reg}}$ such that

$$\lim_{j\to\infty} \|u_j - z_j\|_{L^1(Q_L^+)} = 0, \qquad \lim_{j\to\infty} |\mathscr{F}(u_j) - \mathscr{F}(z_j)| = 0, \tag{3.42}$$

and

$$|A_{u_j}| = m, \quad |B_{u_j}| = M - m. \tag{3.43}$$

We first adjust the volume of $\Omega_{h_{z_j}}$ by a vertical rescaling: namely, we take $\lambda_j := \frac{M}{|\Omega_{h_{z_j}}|}$ (notice that $\lambda_j \to 1$ as $j \to \infty$) and we let $h_j := \lambda_j h_{z_j}$, so that $|\Omega_{h_j}| = M$. We now need to adjust the volume of A_{z_j} and B_{z_j}. Let

$$\widetilde{A}_j := \left\{(x', \lambda_j x_n) \,:\, (x', x_n) \in A_{z_j}\right\}, \qquad \widetilde{B}_j := \left\{(x', \lambda_j x_n) \,:\, (x', x_n) \in B_{z_j}\right\}$$

be the sets obtained by rescaling vertically A_{z_j} and B_{z_j} by the factor λ_j; notice that $\widetilde{A}_j \cup \widetilde{B}_j = \Omega_{h_j}$ and therefore $|\widetilde{A}_j| + |\widetilde{B}_j| = M$. We also remark that, as $\lambda_j \to 1$ and $A_{z_j} \to A_u$, $B_{z_j} \to B_u$ in L^1, we have

$$|\widetilde{A}_j| \to m, \qquad |\widetilde{B}_j| \to M - m \qquad \text{as } j \to \infty.$$

Suppose to fix the ideas that $|\widetilde{A}_j| < m$ (we proceed similarly in the other case). Let $\bar{x} \in \Omega_{h_u}$ be a point of density one for B_u. Since $\widetilde{B}_j \to B_u$ in L^1, we have

$$\lim_{r\to 0^+} \lim_{j\to\infty} \frac{|\widetilde{B}_j \cap B_r(\bar{x})|}{|B_r|} = 1.$$

Hence, it is possible to find $r_0 > 0$ and $j_0 \in \mathbb{N}$ such that

$$\frac{3}{4} \leq \frac{|\widetilde{B}_j \cap B_{r_0}(\bar{x})|}{|B_{r_0}|} \leq 1 \qquad \text{for all } j \geq j_0.$$

Therefore, for every $j \geq j_0$ (for a possibly larger j_0) it is possible to find $r_j \in (0, r_0)$ such that $|\widetilde{B}_j \cap B_{r_j}(\bar{x})| = m - |\widetilde{A}_j| > 0$, since this quantity tends to zero as $j \to \infty$. We eventually define

$$u_j(x', x_n) := \begin{cases} 1 & \text{if } (x', x_n) \in B_{r_j}(\bar{x}) \cap \widetilde{B}_j, \\ z_j(x', x_n/\lambda_j) & \text{if } (x', x_n) \in \Omega_{h_j} \setminus (B_{r_j}(\bar{x}) \cap \widetilde{B}_j), \\ 0 & \text{if } (x', x_n) \in Q_L^+ \setminus \Omega_{h_j}. \end{cases}$$

We then have $h_{u_j} = h_j = \lambda_j h_{z_j}$, so that $u_j \in \mathcal{X}_{\mathrm{reg}}$ and $|\Omega_{h_{u_j}}| = M$. Moreover, $A_{u_j} = \widetilde{A}_j \cup (B_{r_j}(\bar{x}) \cap \widetilde{B}_j)$, hence $|A_{u_j}| = |\widetilde{A}_j| + |\widetilde{B}_j \cap B_{r_j}(\bar{x})| = m$. Thus (3.43) are satisfied. Finally, also the convergences (3.42) hold, since $\lambda_j \to 1$ and $r_j \to 0$. □

4 Regularity of minimizers

In this section we will study the regularity of solutions to the minimum problem

$$\min\bigl\{\overline{\mathscr{F}}(u)\,:\,u\in\mathcal{X},\,|A_u|=m,\,|B_u|=M-m\bigr\},\tag{4.1}$$

whose existence has been established in Theorem 3.5.

The strategy to prove the regularity of minimizers relies, as it is common in these kind of problems, on the regularity theory for area quasi-minimizing clusters (see [22, Part IV] and the references therein). Indeed, we will firstly show in Subsection 4.1 via a penalization technique that it is possible to remove the volume constraint in (4.1) by adding a suitable volume penalization to the functional. Furthermore, the nonlocal part of the energy behaves as a volume-order term thanks to Proposition 2.2. In view of these two properties, it follows that the partition of $\mathbb{R}^n$ given by (A_u,B_u,V_u,S), for a solution u of (4.1), is a quasi-minimizer cluster for the surface energy

$$\mathscr{G}(u):=\int_{J_u\cap Q_L^+}\overline{\Psi}(u^+,u^-)\,\mathrm{d}\mathcal{H}^{n-1},\qquad u\in\mathcal{X}.\tag{4.2}$$

The precise definition of quasi-minimality in our context is given in Definition 4.1 below.

Next, in Subsection 4.2 we exploit the quasi-minimality property to obtain the regularity of minimizers in two dimensions stated in Theorem 1.1. Technical difficulties arise from two fronts: on the one hand, we can only compare with clusters that satisfy the constraint of being the subgraph of a function of bounded variation, a fact that poses a severe restriction on the class of competitors. On the other hand, the interfaces between the phases of the cluster are weighted by different surface tension coefficients. The challenges that arise from these two features prevent us to rely on the standard theory quasi-minimizing clusters, and requires *ad hoc* modifications of the classical proofs. For this reason we develop a regularity theory only in dimension $n=2$, since the general dimensional case requires more refined arguments. We also remark that the regularity properties are obtained under the assumption that the surface tension coefficients satisfy a *strict* triangular inequality (see (4.44)).

4.1 Penalization and quasi-minimality

In this section we show that, in any dimension $n\geq 2$, every solution to the minimum problem (4.1) is a quasi-minimizer for the surface energy (Proposition 4.3), in the sense of the following definition.

Definition 4.1 (Quasi-minimizer). *We say that $u\in\mathcal{X}$ is a* quasi-minimizer *for the surface energy $\mathscr{G}$, defined in (4.2), if there exists $\Lambda>0$ such that for every admissible configuration $v\in\mathcal{X}$ one has*

$$\mathscr{G}(u)\leq\mathscr{G}(v)+\Lambda\bigl(|A_u\triangle A_v|+|B_u\triangle B_v|\bigr).\tag{4.3}$$

We denote, for $\Lambda>0$ and $M>0$, by $\mathcal{A}_{\Lambda,M}$ the class of all configurations $u\in\mathcal{X}$ such that u is a quasi-minimizer for $\mathscr{G}$ with quasi-minimality constant Λ, and $|\Omega_{h_u}|\leq M$.

As a first step we remove the mass constraint in (4.1) by considering a suitable penalized minimum problem, see (4.4). For a discussion of the main idea of the proof, see the Introduction.

Lemma 4.2 (Penalization). *Let $0 < m < M < \infty$. Then there exists $\Lambda > 0$ such that every solution to the constrained minimum problem (4.1) is also a solution to the penalized problem*

$$\min\Big\{\overline{\mathscr{F}}(u) + \Lambda\Big(\big||A_u| - m\big| + \big||\Omega_{h_u}| - M\big|\Big) \,:\, u \in \mathcal{X}\Big\}. \tag{4.4}$$

Proof. Let $u \in \mathcal{X}$ be a minimizer for (4.1), consider a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ with $\lambda_j \to \infty$ as $j \to \infty$, and $u_j \in \mathcal{X}$ solving the minimum problem

$$\min\Big\{\mathscr{H}_{\lambda_j}(v) := \overline{\mathscr{F}}(v) + \lambda_j\Big(\big||A_v| - m\big| + \big||\Omega_{h_v}| - M\big|\Big) \,:\, v \in \mathcal{X}\Big\}, \tag{4.5}$$

whose existence can be shown arguing as in the proof of Theorem 3.5. We will show that, for j large enough, we have

$$|A_{u_j}| = m, \qquad |\Omega_{h_{u_j}}| = M, \tag{4.6}$$

which will imply that u itself is a solution to (4.5) for j large, as desired.

To prove (4.6) we argue by contradiction and we show that, if at least one of the equalities in (4.6) is not satisfied, then for j large enough it is possible to construct by a local variation a configuration $\widetilde{u}_j \in \mathcal{X}$ such that $\mathscr{H}_{\lambda_j}(\widetilde{u}_j) < \mathscr{H}_{\lambda_j}(u_j)$.

The construction of the local variation exploits the same diffeomorphism for both of the mass constraints, applied at different points. In the first part of the proof (Steps 1–4) we thus present the construction of the general diffeomorphism and the corresponding estimates for the change of volume, perimeter and nonlocal energy under this perturbation. To simplify the notation, in the rest of the proof we will write A_j, B_j, and Ω_j in place of A_{u_j}, B_{u_j}, and $\Omega_{h_{u_j}}$ respectively.

Step 1: Definition of the diffeomorphism. We denote by $B'_r := \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$ the $(n-1)$-dimensional ball centered at the origin with radius $r > 0$, and define for $z \in \mathbb{R}^n$

$$C^+(z,r) := z + \big(B'_r \times (0,r)\big), \qquad C^-(z,r) := z + \big(B'_r \times (-r,0)\big),$$

and

$$C(z,r) := C^+(z,r) \cup C^-(z,r) \cup \big(z + B'_r \times \{0\}\big).$$

We next assume that $z = 0$ and we define a family of local perturbations in $C(0,r)$. Precisely, for $|\sigma| < r$ we define the map $\Phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_\sigma(x', x_n) := \begin{cases} \Big(x', x_n + \sigma\big(1 - \frac{|x'|}{r}\big)\big(\frac{x_n}{r} - 1\big)\Big) & \text{if } x \in C^+(0,r), \\ \Big(x', x_n - \sigma\big(1 - \frac{|x'|}{r}\big)\big(\frac{x_n}{r} + 1\big)\Big) & \text{if } x \in C^-(0,r), \\ (x', x_n) & \text{if } x \in \mathbb{R}^n \setminus C(0,r). \end{cases} \tag{4.7}$$

The function Φ_σ is a vertical rescaling with horizontal and vertical cut-off functions. The role of the parameter σ can be seen from Figure 3. Notice that for $|\sigma| < r$ the function Φ_σ is a bi-Lipschitz map and that $\Phi_\sigma(C(0,s)) = C(0,s)$. Moreover, it holds

$$D\Phi(x', x_n) = \left(\begin{array}{c|c} \mathrm{Id}_{n-1} & 0 \\ \hline v_\sigma(x', x_n) & 1 + a_\sigma(x', x_n) \end{array}\right), \tag{4.8}$$

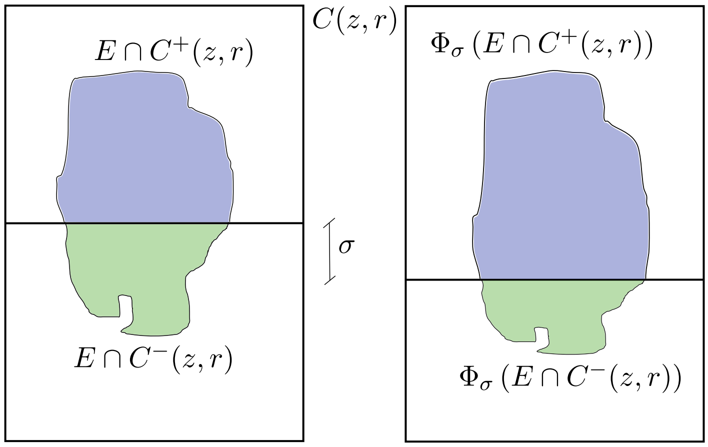

Figure 3: The effect of the map Φ_σ, for $\sigma > 0$, on a set E: it stretches the set on $C^+(z,r)$ and it compresses it on $C^-(z,r)$.

where Id_{n-1} is the $(n-1) \times (n-1)$ identity matrix,

$$a_\sigma(x', x_n) := \begin{cases} \left(1 - \frac{|x'|}{r}\right)\frac{\sigma}{r} & \text{if } (x', x_n) \in C^+(0,r), \\ -\left(1 - \frac{|x'|}{r}\right)\frac{\sigma}{r} & \text{if } (x', x_n) \in C^-(0,r), \end{cases} \tag{4.9}$$

and

$$v_\sigma(x', x_n) := \begin{cases} -\frac{\sigma}{r}\left(\frac{x_n}{r} - 1\right)\frac{x'}{|x'|} & \text{if } (x', x_n) \in C^+(0,r), \\ \frac{\sigma}{r}\left(\frac{x_n}{r} + 1\right)\frac{x'}{|x'|} & \text{if } (x', x_n) \in C^-(0,r). \end{cases} \tag{4.10}$$

When we will perform a perturbation localized in a cylinder centered at a point $z \in \mathbb{R}^n$, we will consider the map $x \mapsto z + \Phi_\sigma(x - z)$.

Step 2: Estimate of the change in volume. Let $E \subset C(0,r)$ be a measurable set. We first estimate the maximal change of volume $||\Phi_\sigma(E)| - |E||$: by using (4.8) and (4.9) we get

$$\begin{aligned} \Big| |\Phi_\sigma(E)| - |E| \Big| &= \frac{|\sigma|}{r} \left| \int_{E \cap C^+(0,r)} \left(1 - \frac{|x'|}{r}\right) \mathrm{d}x - \int_{E \cap C^-(0,r)} \left(1 - \frac{|x'|}{r}\right) \mathrm{d}x \right| \\ &\le \frac{|\sigma|}{r} |E \cap C(0,r)|. \end{aligned} \tag{4.11}$$

Next, we prove more refined estimates on the change of volume of a set E in the upper and lower cylinders $C^+(0,r)$, $C^-(0,r)$. We first consider the case $\sigma > 0$. In this case, the followings hold:

(i) For every $\varepsilon > 0$ and $\sigma \in (0,r)$, if $|E \cap C^+(0,r)| < \varepsilon r^n$ then

$$0 \le |\Phi_\sigma(E \cap C^+(0,r))| - |E \cap C^+(0,r)| \le U(\varepsilon)|\sigma| r^{n-1}, \tag{4.12}$$

where

$$U(\varepsilon) := \left[1 - \frac{n-1}{n}\left(\frac{\varepsilon}{\omega_{n-1}}\right)^{\frac{1}{n-1}}\right]\varepsilon.$$

(ii) For every $\mu \in (0, \omega_{n-1})$ and $\sigma \in (0, r)$, if $|E \cap C^-(0,r)| > \mu r^n$, then

$$|\Phi_\sigma(E \cap C^-(0,r))| - |E \cap C^-(0,r))| \leq -L(\mu)|\sigma| r^{n-1} < 0, \tag{4.13}$$

where

$$L(\mu) := \left[\frac{1}{n} - \left(1 - \frac{\mu}{\omega_{n-1}}\right) + \frac{n-1}{n}\left(1 - \frac{\mu}{\omega_{n-1}}\right)^{\frac{n}{n-1}} \right] \omega_{n-1}.$$

In (4.12)–(4.13) we have written $|\sigma|$ in place of σ to stress the fact that the same estimates hold also in the case $\sigma < 0$ up to exchanging the roles of $C^+(0,r)$ and $C^-(0,r)$, as can be easily checked. Notice that $U(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that $L(\mu)$ is strictly positive, and more precisely $L(\mu) \in (0, \frac{\omega_{n-1}}{n})$ for every choice of $\mu \in (0, \omega_{n-1})$, with $L(\mu) \to 0$ as $\mu \to 0$, $L(\mu) \to \frac{\omega_{n-1}}{n}$ as $\mu \to \omega_{n-1}$.

To prove (4.12) we notice that by (4.8) and (4.9), and since $\sigma > 0$,

$$|\Phi_\sigma(E \cap C^+(0,r))| - |E \cap C^+(0,r)| = \frac{\sigma}{r} \int_{E \cap C^+(0,r)} \left(1 - \frac{|x'|}{r}\right) \,\mathrm{d}x \leq \frac{\sigma}{r} \int_{F_\varepsilon} \left(1 - \frac{|x'|}{r}\right) \,\mathrm{d}x,$$

where

$$F_\varepsilon := B'_{r_\varepsilon} \times (0,r), \qquad \text{with} \quad r_\varepsilon := r(\varepsilon/\omega_{n-1})^{\frac{1}{n-1}}.$$

By a direct computation we get (4.12). To prove (4.13), we similarly estimate

$$|\Phi_\sigma(E \cap C^-(0,r))| - |E \cap C^-(0,r)| = -\frac{\sigma}{r} \int_{E \cap C^-(0,r)} \left(1 - \frac{|x'|}{r}\right) \,\mathrm{d}x \leq -\frac{\sigma}{r} \int_{G_\mu} \left(1 - \frac{|x'|}{r}\right) \,\mathrm{d}x,$$

where

$$G_\mu := C^-(0,r) \backslash (B'_{s_\mu} \times (-r, 0)), \qquad \text{with} \quad s_\mu := r(1 - \mu/\omega_{n-1})^{\frac{1}{n-1}}.$$

We conclude again by a direct computation.

Step 3: Estimate of the change in perimeter. Given a countably $\mathcal{H}^{n-1}$-rectifiable set $\Sigma \subset \mathbb{R}^n$, by the generalized area formula (see [4, Theorem 2.91]) we have that

$$\mathcal{H}^{n-1}(\Phi_\sigma(\Sigma)) - \mathcal{H}^{n-1}(\Sigma) = \int_\Sigma \left(J_{n-1} \mathrm{d}_x^\Sigma \Phi_\sigma - 1\right) \,\mathrm{d}\mathcal{H}^{n-1}(x), \tag{4.14}$$

where $\mathrm{d}_x^\Sigma \Phi_\sigma : \pi_x^\Sigma \to \mathbb{R}^n$ denotes the tangential differential of Φ_σ at $x \in \Sigma$ along the approximate tangent space π_x^Σ to Σ, and the area factor $J_{n-1} \mathrm{d}_x^\Sigma \Phi_\sigma$ is defined as (see [4, Definition 2.68])

$$J_{n-1} \mathrm{d}_x^\Sigma \Phi_\sigma := \sqrt{\det\left((\mathrm{d}_x^\Sigma \Phi_\sigma)^* \circ \mathrm{d}_x^\Sigma \Phi_\sigma\right)} \tag{4.15}$$

(here $(\mathrm{d}_x^\Sigma \Phi_\sigma)^*$ is the adjoint of the linear map $\mathrm{d}_x^\Sigma \Phi_\sigma$).

In order to estimate (4.15), fix $x \in \Sigma$ and let $\tau_1, \ldots, \tau_{n-1}$ be an orthonormal basis for the approximate tangent space π_x^Σ. By using (4.8), (4.9), and (4.10), for all $i, j \in \{1, \ldots, n-1\}$ we have

$$\begin{aligned}
\left((\mathrm{d}_x^\Sigma \Phi_\sigma)^* \circ \mathrm{d}_x^\Sigma \Phi_\sigma\right)_{ij} &= \sum_{k=1}^n \left(\nabla \Phi_\sigma^k(x) \cdot \tau_i\right)\left(\nabla \Phi_\sigma^k(x) \cdot \tau_j\right) \\
&= \tau_i \cdot \tau_j + \tau_i^n(\tau_j \cdot w_\sigma(x)) + \tau_j^n(\tau_i \cdot w_\sigma(x)) + (\tau_i \cdot w_\sigma(x))(\tau_j \cdot w_\sigma(x)),
\end{aligned}$$

where $\Phi_\sigma = (\Phi_\sigma^1, \dots, \Phi_\sigma^n)$ and $\tau_i = (\tau_i^1, \dots, \tau_i^n)$ denote the components with respect to the canonical base of $\mathbb{R}^n$, and $w_\sigma(x) \coloneqq (v_\sigma(x), a_\sigma(x))$. By using the fact that $|w_\sigma(x)| \le \sqrt{2}|\sigma|/r$ and the general formula $\det(I + tA) = 1 + t\,\mathrm{trace}(A) + O(t^2)$ as $t \to 0$, we get

$$\det\big((\mathrm{d}_x^\Sigma \Phi_\sigma)^* \circ \mathrm{d}_x^\Sigma \Phi_\sigma\big) = 1 + 2\sum_{i=1}^{n-1} \tau_i^n (\tau_i \cdot w_\sigma(x)) + O\Big(\Big(\frac{\sigma}{r}\Big)^2\Big),$$

where $O\big(\big(\frac{\sigma}{r}\big)^2\big) \le C\big(\frac{\sigma}{r}\big)^2$ for a constant $C > 0$ independent of σ, r, and of $x \in \Sigma$. Therefore by (4.14) we find for $|\sigma|$ sufficiently small

$$\big|J_{n-1}\mathrm{d}_x^\Sigma \Phi_\sigma - 1\big| = \left|\sqrt{\det\big((\mathrm{d}_x^\Sigma \Phi_\sigma)^* \circ \mathrm{d}_x^\Sigma \Phi_\sigma\big)} - 1\right| \le c_0 \frac{|\sigma|}{r},$$

where $c_0 > 0$ is a dimensional constant. This, together with (4.14), yields

$$\big|\,\mathcal{H}^{n-1}(\Phi_\sigma(\Sigma)) - \mathcal{H}^{n-1}(\Sigma)\,\big| \le c_0 \frac{|\sigma|}{r} \mathcal{H}^{n-1}(\Sigma). \tag{4.16}$$

Step 4: Estimate of the change of the nonlocal term. Finally, we estimate the change in the nonlocal energy. We note that, by Proposition 2.2, it is enough to get an estimate on $|\Phi_\sigma(E) \triangle E|$ for a general set E with finite perimeter. As in [1, Proposition 2.7], we prove the estimate by approximating the characteristic function of a set by a sequence of smooth functions.

From the definition (4.7) of Φ_σ, it follows that we can write $\Phi_\sigma^{-1}(x', x_n) = (x', x_n + \phi_\sigma(x_n))$ with $|\phi_\sigma(x_n)| \le |\sigma|$. Then for $f \in C^1(\mathbb{R}^n)$ we have

$$\begin{aligned}
\int_{C(0,r)} |f - f \circ \Phi_\sigma^{-1}| \,\mathrm{d}x &= \int_{B_r'} \int_{-r}^{r} \big|f(x', x_n) - f\left(x', x_n + \phi_\sigma(x_n)\right)\big| \,\mathrm{d}x_n \,\mathrm{d}x' \\
&= \int_{B_r'} \int_{-r}^{r} \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\big(x', x_n + t\phi_\sigma(x_n)\big) \,\mathrm{d}t \right| \mathrm{d}x_n \,\mathrm{d}x' \\
&\le \int_{B_r'} \int_{-r}^{r} \left| \int_0^1 \frac{\partial f}{\partial x_n}\big(x', x_n + t\phi_\sigma(x_n)\big) \,\mathrm{d}t \right| |\phi_\sigma(x_n)| \,\mathrm{d}x_n \,\mathrm{d}x' \\
&\le |\sigma| \int_{B_r'} \int_{-r}^{r} \int_0^1 \left| \frac{\partial f}{\partial x_n}\big(x', x_n + t\phi_\sigma(x_n)\big) \right| \mathrm{d}t \,\mathrm{d}x_n \,\mathrm{d}x' \\
&\le |\sigma| \int_{C(0,r)} |\nabla f(x)| \,\mathrm{d}x.
\end{aligned} \tag{4.17}$$

Let now $E \subset \mathbb{R}^n$ be a set with finite perimeter and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of smooth functions such that $f_k \to \chi_E$ in L^1 and $\|\nabla f_k\|_{L^1} \to \mathscr{P}(E)$. Then also $f_k \circ \Phi_\sigma^{-1} \to \chi_E \circ \Phi_\sigma^{-1}$ in L^1. Therefore applying (4.17) to the function f_k and passing to the limit as $k \to \infty$ yields

$$|\Phi_\sigma(E) \triangle E| = \int_{C(0,r)} \big|\chi_E - \chi_E \circ \Phi_\sigma^{-1}\big| \,\mathrm{d}x \le |\sigma| \mathscr{P}(E). \tag{4.18}$$

Step 5: General strategy. We can now go back to the main argument of the proof and show that any solution u_j of the penalized problem (4.5) satisfies the mass constraints (4.6), for j large enough.

The idea of the proof is to assume by contradiction that one of the mass constraints in (4.6) is not satisfied, and to construct a perturbation of u_j in a cylinder $C(z,r)$ by means of the maps Φ_{σ_j}. More precisely, we will choose a point $z \in \mathbb{R}^n$, a radius $r > 0$ and scaling coefficients $\sigma_j \in (-r, r)$ and define

$$\widetilde{u}_j(x) \coloneqq u_j(z + \Phi_{\sigma_j}^{-1}(x - z)). \tag{4.19}$$

This is a local perturbation inside $C(z,r)$ (the center and the radius will be chosen in such a way that the cylinder does not intersect the substrate S) such that the phases of the new configuration $\widetilde{u}_j$ are given by

$$\widetilde{A}_j = z + \Phi_{\sigma_j}(A_j - z), \qquad \widetilde{B}_j = z + \Phi_{\sigma_j}(B_j - z), \qquad \widetilde{\Omega}_j = z + \Phi_{\sigma_j}(\Omega_j - z).$$

Thanks to (4.16), (4.18) and Proposition 2.2, we get the estimate

$$\begin{aligned} \mathscr{H}_{\lambda_j}(\widetilde{u}_j) - \mathscr{H}_{\lambda_j}(u_j) \leq \Big(\frac{\tilde{c}_0}{r} + \gamma L_{\mathcal{N}}\Big)\big(\mathscr{P}(A_j) + \mathscr{P}(B_j)\big)|\sigma_j| \\ + \lambda_j\Big(\big||\widetilde{A}_j| - m\big| + \big||\widetilde{\Omega}_j| - M\big| - \big||A_j| - m\big| - \big||\Omega_j| - M\big|\Big). \end{aligned} \tag{4.20}$$

where $\tilde{c_0}$ depends on the constant c_0 in (4.16) and on the surface tension coefficients. The goal would be then to show that, if at least one of the volume constraints is not satisfied, then it is possible to choose z, r and σ_j so that

$$\big||\widetilde{A}_j| - m\big| + \big||\widetilde{\Omega}_j| - M\big| - \big||A_j| - m\big| - \big||\Omega_j| - M\big| \leq -C|\sigma_j|r^{n-1}, \tag{4.21}$$

for some $C > 0$ independent of j. As $\lambda_j \to \infty$, the combination of (4.20) and (4.21) shows that $\mathscr{H}_{\lambda_j}(\widetilde{u}_j) < \mathscr{H}_{\lambda_j}(u_j)$ for j large enough, which is a contradiction with the minimality of u_j in (4.5).

In the next two steps we will implement the previous strategy. We first observe that, by using u as a competitor in the minimum problem (4.5) and since $\overline{\mathscr{F}}(u) < \infty$, we obtain the bounds

$$\sup_{j\in\mathbb{N}}\Big(\mathscr{P}(A_j) + \lambda_j\big||A_j| - m\big|\Big) < \infty, \quad \sup_{j\in\mathbb{N}}\Big(\mathscr{P}(B_j) + \lambda_j\big||B_j| - (M-m)\big|\Big) < \infty. \tag{4.22}$$

Thus, up to a subsequence (not relabeled), we get that $A_j \to A$ and $B_j \to B$ in L^1, with $|A| = m$, $|B| = M - m$ since $\lambda_j \to \infty$. We also have $\Omega_j \to \Omega \coloneqq A \cup B$. Notice that Ω is still the subgraph of an admissible profile.

In the following, given a point $x_0 \in \mathbb{R}^n$, $r > 0$, and a direction $\nu = (\nu', \nu_n) \in \mathbb{S}^{n-1}$ with $\nu_n \neq 0$, we define

$$y_r \coloneqq x_0 + \left(r \cos\left(\arctan\frac{|\nu_n|}{|\nu'|}\right)\right)\nu \tag{4.23}$$

and we consider the corresponding cylinder $C(y_r, r)$. The choice of the point y_r guarantees, since $\nu_n \neq 0$, that there exists a constant $c_\nu > 0$, independent of r, such that if $\nu_n > 0$

$$C^+(y_r, r) \cap \{(x - x_0)\cdot\nu < 0\} = \emptyset, \qquad \big|C^-(y_r, r) \cap \{(x - x_0)\cdot\nu < 0\}\big| = 2c_\nu r^n, \tag{4.24}$$

while if $\nu_n < 0$

$$C^-(y_r, r) \cap \{(x - x_0)\cdot\nu < 0\} = \emptyset, \qquad \big|C^+(y_r, r) \cap \{(x - x_0)\cdot\nu < 0\}\big| = 2c_\nu r^n. \tag{4.25}$$

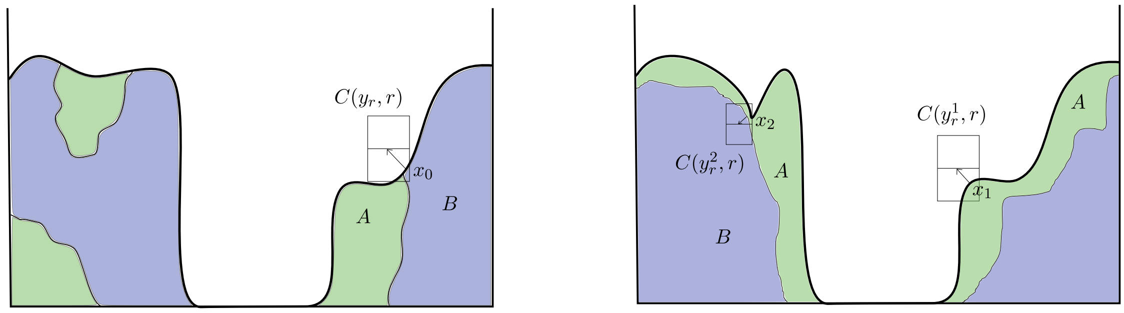

Figure 4: The constructions we employ in Step 6 in order to prove that $|\Omega_j| = M$. On the left the first case, where we can locally modify the graph by changing by little the volume of A_j. When this is not possible, we use another local modification to adjust the volume of A_j, as can be seen on the figure on the right.

Notice that the strict positivity of c_ν is a consequence of the fact that $\nu_n \neq 0$.

Step 6: Fixing the total volume. Assume by contradiction that $|\Omega_j| \neq M$ for infinitely many j. We will consider for simplicity the case $|\Omega_j| > M$ for all j, as the other case can be treated by a similar argument.

Case 1. Assume that there exists $x_0 \in \partial^*\Omega \cap \partial^* B$ such that $\nu_\Omega(x_0) \cdot e_n > 0$ (where ν_Ω denotes the exterior normal). We consider the point y_r and the constant c_ν defined in (4.23) and (4.24) respectively, for $\nu = \nu_\Omega(x_0)$ and $r > 0$ to be chosen later.

De Giorgi's structure theorem for sets of finite perimeter ([4, Theorem 3.59]) together with (4.24) ensures that

$$\lim_{r\to 0} \frac{|\Omega \cap C^+(y_r,r)|}{r^n} = \lim_{r\to 0} \frac{|A \cap C(y_r,r)|}{r^n} = 0, \qquad \lim_{r\to 0} \frac{|\Omega \cap C^-(y_r,r)|}{r^n} = 2c_\nu.$$

Therefore, for every $\varepsilon > 0$, the fact that $\chi_{\Omega_j} \to \chi_\Omega$, $\chi_{A_j} \to \chi_A$ and $\chi_{B_j} \to \chi_B$ in L^1 yields the existence of $r \in (0,1)$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ the following holds (see Figure 4 on the left):

$$|\Omega_j \cap C^+(y_r,r)| < \varepsilon r^n, \qquad |\Omega_j \cap C^-(y_r,r)| > c_\nu r^n, \tag{4.26}$$

$$|A_j \cap C(y_r,r)| < \varepsilon r^n. \tag{4.27}$$

Moreover, for r small enough we can also guarantee that the cylinder $C(y_r,r)$ is contained in the upper half-space and does not intersect the substrate. We then choose $\sigma_j > 0$ and consider the perturbation defined in (4.19) centered at the point $z = y_r$. In view of (4.26), by using (4.12) and (4.13), we get

$$|\widetilde{\Omega}_j| - |\Omega_j| \leq -\big(L(c_\nu) - U(\varepsilon)\big)\sigma_j r^{n-1}.$$

On the other hand, by (4.11) and (4.27),

$$\big||\widetilde{A}_j| - |A_j|\big| \leq \frac{\sigma_j}{r}|A_j \cap C(y_r,r)| \leq \varepsilon\sigma_j r^{n-1}.$$

Therefore, noting that we can assume $M < |\widetilde{\Omega}_j| < |\Omega_j|$ (it is sufficient to choose σ_j and ε small enough), we find

$$\big||\widetilde{\Omega}_j| - M\big| - \big||\Omega_j| - M\big| + \big||\widetilde{A}_j| - m\big| - \big||A_j| - m\big| \leq |\widetilde{\Omega}_j| - |\Omega_j| + \big||\widetilde{A}_j| - |A_j|\big|$$

$$\le -\big(L(c_\nu) - U(\varepsilon) - \varepsilon\big)\sigma_j r^{n-1}.$$

By choosing ε sufficiently small, we can ensure that $U(\varepsilon)+\varepsilon < L(c_\nu)$, which yields (4.21) and leads to the desired contradiction in this case.

Case 2. If the assumption of the previous case does not hold, we can find a point $x_1 \in \partial^*\Omega \cap \partial^* A$ such that $\nu_\Omega(x_1)\cdot e_n > 0$. Since $0 < m < M$, it is possible to find a second point $x_2 \in \partial^* A \cap \partial^* B$ such that $\nu_A(x_2)\cdot e_n < 0$.

We will consider the composition of two perturbations of the form (4.19) localized in two disjoint cylinders $C(y_r^1, r)$ and $C(y_r^2, r)$, where (see also (4.23))

$$y_r^1 := x_1 + \left[r\,\cos\left(\arctan\frac{|(\nu_\Omega(x_1))_n|}{|(\nu_\Omega(x_1))'|}\right)\right]\nu_\Omega(x_1),$$
$$y_r^2 := x_2 + \left[r\,\cos\left(\arctan\frac{|(\nu_A(x_2))_n|}{|(\nu_A(x_2))'|}\right)\right]\nu_A(x_2).$$

Let

$$E_r^1 := \{(x - x_1)\cdot \nu_\Omega(x_1) < 0\} \cap C(y_r^1, r), \qquad E_r^2 := \{(x - x_2)\cdot \nu_A(x_2) < 0\} \cap C(y_r^2, r).$$

Note that $E_r^1 \subset C^-(y_r^1, r)$ and that $E_r^2 \subset C^+(y_r^2, r)$. We let $\mu := c_{\nu_\Omega(x_1)}$, where the constant c_ν, for a vector ν, is defined in (4.24). As in the previous case, fixed $\varepsilon > 0$, we can find $r > 0$ and $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$ we have (see Figure 4 on the right)

$$|\Omega_j \cap C^+(y_r^1, r)| < \varepsilon r^n, \qquad |\Omega_j \cap C^-(y_r^1, r)| > \mu r^n, \tag{4.28}$$

$$|V_j \cap C(y_r^2, r)| < \varepsilon r^n, \tag{4.29}$$

and

$$|\big(A_j \cap C(y_r^1, r)\big) \triangle E_r^1| < \varepsilon r^n, \qquad |\big(A_j \cap C(y_r^2, r)\big) \triangle E_r^2| < \varepsilon r^n. \tag{4.30}$$

By reducing the value of $r > 0$ we can further assume that the two cylinders $C(y_r^1, r)$ and $C(y_r^2, r)$ are disjoint and do not intersect the substrate. For a fixed sequence $\{\sigma_j^1\}_{j\in\mathbb{N}} \subset (0, r)$, we define a second sequence $\{\sigma_j^2\}_{j\in\mathbb{N}}$ as

$$\sigma_j^2 := \alpha\sigma_j^1, \qquad \text{where } \alpha := \frac{\int_{E_r^1 - y_r^1}\big(1 - \frac{|x'|}{r}\big)\,\mathrm{d}x}{\int_{E_r^2 - y_r^2}\big(1 - \frac{|x'|}{r}\big)\,\mathrm{d}x} \tag{4.31}$$

for each $j \in \mathbb{N}$. Notice that α is independent of r by scale invariance. Then, we consider the configuration $\widetilde{u}_j$ obtained by applying to u_j the composition of the two perturbations $y_r^1 + \Phi_{\sigma_j^1}(\cdot - y_r^1)$ and $y_r^2 + \Phi_{\sigma_j^2}(\cdot - y_r^2)$. We denote the sets of the new partition determined by $\widetilde{u}_j$ by $\widetilde{A}_j$, $\widetilde{B}_j$, $\widetilde{\Omega}_j = \widetilde{A}_j \cup \widetilde{B}_j$, $\widetilde{V}_j$.

We first consider the variation of the volume of Ω_j. By (4.28), and since $\sigma_j^1 > 0$, we can apply (4.12) and (4.13) and obtain

$$|\widetilde{\Omega}_j \cap C(y_r^1, r)| - |\Omega_j \cap C(y_r^1, r)| \le -\big(L(\mu) - U(\varepsilon)\big)\sigma_j^1 r^{n-1}.$$

On the other hand, by (4.29) and using (4.11) we find

$$|\widetilde{\Omega}_j \cap C(y_r^2, r)| - |\Omega_j \cap C(y_r^2, r)| = |V_j \cap C(y_r^2, r)| - |\widetilde{V}_j \cap C(y_r^2, r)|$$
$$\le \frac{\sigma_j^2}{r}|V_j \cap C(y_r^2, r)| \le \varepsilon\sigma_j^2 r^{n-1}.$$

By combining the two estimates and recalling (4.31) it follows that

$$|\widetilde{\Omega}_j| - |\Omega_j| \le -\Big(L(\mu) - U(\varepsilon) - \alpha\varepsilon\Big)\sigma_j^1 r^{n-1}. \tag{4.32}$$

Next, we look at the variation of the volume of A_j. We have for $i = 1, 2$

$$|\widetilde{A}_j \cap C(y_r^i, r)| - |A_j \cap C(y_r^i, r)| = |\Phi_{\sigma_j^i}(E_r^i - y_r^i)| - |E_r^i| + R_j^i, \tag{4.33}$$

where thanks to (4.30) and (4.11)

$$|R_j^i| \le \big||\Phi_{\sigma_j^i}((A_j \cap C(y_r^i, r))\triangle E_r^i - y_r^i)| - |(A_j \cap C(y_r^i, r))\triangle E_r^i|\big| \le \varepsilon\sigma_j^i r^{n-1}. \tag{4.34}$$

Also notice that the choice of σ_j^2 in (4.31) guarantees exactly that

$$\begin{aligned}\Big(|\Phi_{\sigma_j^1}(E_r^1 - y_r^1)| - |E_r^1|\Big) &+ \Big(|\Phi_{\sigma_j^2}(E_r^2 - y_r^2)| - |E_r^2|\Big) \\ &= -\frac{\sigma_j^1}{r}\int_{E_r^1 - y_r^1}\Big(1 - \frac{|x'|}{r}\Big)\,\mathrm{d}x + \frac{\sigma_j^2}{r}\int_{E_r^2 - y_r^2}\Big(1 - \frac{|x'|}{r}\Big)\,\mathrm{d}x = 0.\end{aligned} \tag{4.35}$$

Therefore, from (4.33), (4.34), and (4.35), we get

$$\begin{aligned}\big||\widetilde{A}_j| - |A_j|\big| &= \big||\widetilde{A}_j \cap C(y_r^1, r)| - |A_j \cap C(y_r^1, r)| + |\widetilde{A}_j \cap C(y_r^2, r)| - |A_j \cap C(y_r^2, r)|\big| \\ &\le \big||\Phi_{\sigma_j^1}(E_r^1 - y_r^1)| - |E_r^1| + |\Phi_{\sigma_j^2}(E_r^2 - y_r^2)| - |E_r^2|\big| + |R_j^1| + |R_j^2| \\ &\le \varepsilon(\sigma_j^1 + \sigma_j^2)r^{n-1} = \varepsilon(1+\alpha)\sigma_j^1 r^{n-1}.\end{aligned} \tag{4.36}$$

We can now conclude as follows. Similarly to (4.20) we find

$$\begin{aligned}\mathscr{H}_{\lambda_j}(\tilde{u}_j) - \mathscr{H}_{\lambda_j}(u_j) &\le \Big(\frac{\tilde{c}_0}{r} + \gamma L_{\mathcal{N}}\Big)\big(\mathscr{P}(A_j) + \mathscr{P}(B_j)\big)(\sigma_j^1 + \sigma_j^2) \\ &\quad + \lambda_j\Big(\big||\widetilde{A}_j| - m\big| + \big||\widetilde{\Omega}_j| - M\big| - \big||A_j| - m\big| - \big||\Omega_j| - M\big|\Big) \\ &\le C(1+\alpha)\sigma_j^1 + \lambda_j\Big(\big||\widetilde{A}_j| - |A_j|\big| + |\widetilde{\Omega}_j| - |\Omega_j|\Big) \\ &\le \Big[C(1+\alpha) - \lambda_j\Big(L(\mu) - U(\varepsilon) - \alpha\varepsilon - (1+\alpha)\varepsilon\Big)r^{n-1}\Big]\sigma_j^1\end{aligned}$$

where we used (4.22) in the second inequality, and (4.32), (4.36) in the last one. We can therefore choose $\varepsilon > 0$ small enough so that the constant multiplying λ_j is strictly negative; as $\lambda_j \to +\infty$, this provides the desired contradiction with the minimality of u_j.

Step 7: Fixing the volume of each phase. In this step we conclude the proof by showing that $|A_j| = m$ for j large. Thanks to the previous step, we can assume that $|\Omega_j| = M$ for all $j \in \mathbb{N}$. Suppose by contradiction that $A_j \neq m$ for infinitely many j. We consider for simplicity only the case $|A_j| > m$ for all j, as the other case can be treated with similar computations.

Case 1. Assume that there exists $x_0 \in \partial^* A \cap \partial^* B$ such that $\nu_A(x_0) \cdot e_n \neq 0$. We assume to fix the ideas to be in the case $\nu_A(x_0) \cdot e_n > 0$; in the other case, it is sufficient to exchange the roles of the upper and lower cylinders in the computations below. We consider, for $r > 0$ to be chosen, the point y_r and the constant c_ν defined in (4.23) and (4.24) respectively, corresponding to $\nu = \nu_A(x_0)$.

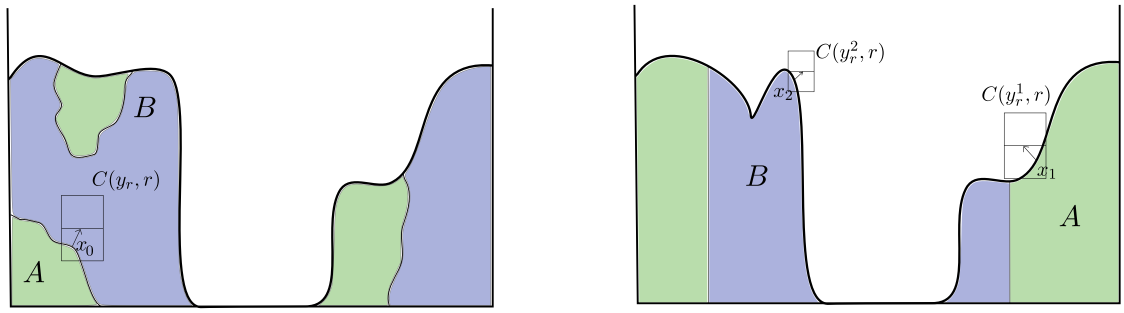

Figure 5: The constructions we employ in Step 7 in order to prove that $|A_j| = m$. On the left the first case, where we can locally modify the set A_j by changing by little the volume of Ω_j. When this is not possible by keeping the constraint of being a graph, we use another local modification to adjust the volume of Ω_j, as can be seen on the figure on the right.

Fix $\varepsilon > 0$. By De Giorgi's structure theorem and the convergence $\chi_{A_j} \to \chi_A$, there exist $r > 0$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ it holds (see Figure 5 on the left)

$$|A_j \cap C^+(y_r, r)| < \varepsilon r^n, \qquad |A_j \cap C^-(y_r, r)| > c_\nu r^n, \tag{4.37}$$

and

$$|V_j \cap C(y_r, r)| < \varepsilon r^n. \tag{4.38}$$

Moreover, for r small enough we can also guarantee that the cylinder $C(y_r, r)$ is contained in the upper half-space and does not intersect the substrate. We then choose $\sigma_j > 0$ and consider the perturbation defined in (4.19) centered at the point $z = y_r$. From (4.37), (4.12), and (4.13), we have

$$|\widetilde{A}_j| - |A_j| \leq -(L(c_\nu) - U(\varepsilon))\sigma_j r^{n-1}.$$

Moreover, from (4.38) and (4.11) we can estimate

$$\big||\widetilde{V}_j \cap C(y_r, r)| - |V_j \cap C(y_r, r)|\big| \leq \varepsilon \sigma_j r^{n-1}.$$

Thus, using the fact that $|\Omega_j| = M$ for all $j \in \mathbb{N}$, and that $m < |\widetilde{A}_j| < |A_j|$ (by choosing σ_j and ε small enough), we obtain

$$\begin{aligned}
\big||\widetilde{A}_j| - m\big| + \big||\widetilde{\Omega}_j| - M\big| - \big||A_j| - m\big| - \big||\Omega_j| - M\big| \\
&= \big||\widetilde{\Omega}_j| - |\Omega_j|\big| + |\widetilde{A}_j| - |A_j| \\
&= \big||\widetilde{V}_j \cap C(y_r, r)| - |V_j \cap C(y_r, r)|\big| + |\widetilde{A}_j| - |A_j| \\
&\leq -(L(c_\nu) - U(\varepsilon) - \varepsilon)\sigma_j r^{n-1}.
\end{aligned}$$

Therefore, by choosing $\varepsilon > 0$ small enough we get (4.21), as desired.

Case 2. Finally, assume that $\nu_A(x) \cdot e_n = 0$ for all $x \in \partial^* A \cap \partial^* B$. The construction in this case is similar to the one in Step 6, Case 2. Since $0 < m < M$, we get that there exist $x_1 \in \partial^* A \cap \partial^* V$ and $x_2 \in \partial^* B \cap \partial^* V$. We let, for $r > 0$, y_r^1 and y_r^2 be the points defined by (4.23) corresponding to the choice of $\nu_\Omega(x_1)$ and $\nu_\Omega(x_2)$, respectively, and

$$E_r^1 := \{(x - x_1) \cdot \nu_\Omega(x_1) < 0\} \cap C(y_r^1, r), \qquad E_r^2 := \{(x - x_2) \cdot \nu_\Omega(x_2) < 0\} \cap C(y_r^2, r).$$

We let $\mu := c_{\nu_\Omega(x_1)} > 0$, where the constant c_ν, for a vector ν, is defined in (4.24). As in the previous cases, for fixed $\varepsilon > 0$, we can find $r > 0$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have (see Figure 5 on the right)

$$|\big(\Omega_j \cap C(y_r^1, r)\big) \triangle E_r^1| < \varepsilon r^n, \qquad |A_j \cap C^-(y_r^1, r)| > \mu r^n, \qquad |A_j \cap C^+(y_r^1, r)| < \varepsilon r^n \tag{4.39}$$

and

$$|\big(\Omega_j \cap C(y_r^2, r)\big) \triangle E_r^2| < \varepsilon r^n, \qquad |A_j \cap C(y_r^2, r)| < \varepsilon r^n. \tag{4.40}$$

By reducing the value of $r > 0$ we can further assume that the two cylinders $C(y_r^1, r)$ and $C(y_r^2, r)$ are disjoint and do not intersect the substrate. For a fixed sequence $\{\sigma_j^1\}_{j\in\mathbb{N}} \subset (0, r)$, we define a second sequence $\{\sigma_j^2\}_{j\in\mathbb{N}}$ as

$$\sigma_j^2 := -\alpha\sigma_j^1, \qquad \text{where } \alpha := \frac{\int_{E_r^1 - y_r^1} \big(1 - \frac{|x'|}{r}\big)\,\mathrm{d}x}{\int_{E_r^2 - y_r^2} \big(1 - \frac{|x'|}{r}\big)\,\mathrm{d}x} \tag{4.41}$$

for each $j \in \mathbb{N}$. Notice that $\sigma_j^2 < 0$ and that α is independent of r by scale invariance. Then, we consider the configuration $\widetilde{u}_j$ obtained by applying to u_j the composition of the perturbations $y_r^1 + \Phi_{\sigma_j^1}(\cdot - y_r^1)$, $y_r^2 + \Phi_{\sigma_j^2}(\cdot - y_r^2)$. We denote the sets of the new partition determined by $\widetilde{u}_j$ by $\widetilde{A}_j$, $\widetilde{B}_j$, $\widetilde{\Omega}_j = \widetilde{A}_j \cup \widetilde{B}_j$, $\widetilde{V}_j$.

We first consider the variation of the volume of A_j. By using the last two inequalities in (4.39) and the last inequality in (4.40), together with (4.12), (4.13), (4.11), we get

$$|\widetilde{A}_j| - |A_j| \leq -\big(L(\mu) - U(\varepsilon)\big)\sigma_j^1 r^{n-1} + \varepsilon|\sigma_j^2| r^{n-1}. \tag{4.42}$$

The choice (4.41) guarantees that

$$\begin{aligned}\Big(|\Phi_{\sigma_j^1}(E_r^1 - y_r^1)| - |E_r^1|\Big) + \Big(|\Phi_{\sigma_j^2}(E_r^2 - y_r^2)| - |E_r^2|\Big)& \\ = -\frac{\sigma_j^1}{r}\int_{E_r^1 - y_r^1}\Big(1 - \frac{|x'|}{r}\Big)\,\mathrm{d}x - \frac{\sigma_j^2}{r}\int_{E_r^2 - y_r^2}\Big(1 - \frac{|x'|}{r}\Big)\,\mathrm{d}x = 0,&\end{aligned}$$

therefore by arguing as in (4.36) we find

$$\begin{aligned}\big||\widetilde{\Omega}_j| - |\Omega_j|\big| &= \big||\widetilde{\Omega}_j \cap C(y_r^1, r)| - |\Omega_j \cap C(y_r^1, r)| + |\widetilde{\Omega}_j \cap C(y_r^2, r)| - |\Omega_j \cap C(y_r^2, r)|\big| \\ &\leq \big||\Phi_{\sigma_j^1}(E_r^1 - y_r^1)| - |E_r^1| + |\Phi_{\sigma_j^2}(E_r^2 - y_r^2)| - |E_r^2|\big| + |R_j^1| + |R_j^2| \\ &\leq \varepsilon(\sigma_j^1 + |\sigma_j^2|)r^{n-1} = \varepsilon(1 + \alpha)\sigma_j^1 r^{n-1}.\end{aligned} \tag{4.43}$$

Therefore, similarly to (4.20) we find

$$\begin{aligned}\mathscr{H}_{\lambda_j}(\widetilde{u}_j) - \mathscr{H}_{\lambda_j}(u_j) &\leq \Big(\frac{\tilde{c}_0}{r} + \gamma L_{\mathcal{N}}\Big)\big(\mathscr{P}(A_j) + \mathscr{P}(B_j)\big)(\sigma_j^1 + |\sigma_j^2|) \\ &\qquad + \lambda_j\Big(\big||\widetilde{A}_j| - m\big| + \big||\widetilde{\Omega}_j| - M\big| - \big||A_j| - m\big| - \big||\Omega_j| - M\big|\Big) \\ &\leq C(1 + \alpha)\sigma_j^1 + \lambda_j\Big(\big||\widetilde{\Omega}_j| - |\Omega_j|\big| + |\widetilde{A}_j| - |A_j|\Big) \\ &\leq \Big[C(1 + \alpha) - \lambda_j\Big(L(\mu) - U(\varepsilon) - \alpha\varepsilon - (1 + \alpha)\varepsilon\Big)r^{n-1}\Big]\sigma_j^1.\end{aligned}$$

where we used (4.22) in the second inequality, and (4.42), (4.43) in the last one. We can therefore choose $\varepsilon > 0$ small enough so that the constant multiplying λ_j is strictly negative; as $\lambda_j \to +\infty$, this provides the desired contradiction with the minimality of u_j. □

Proposition 4.3. *Let u be a solution to the minimum problem* (4.1)*. Then u is a quasi-minimizer for the surface energy $\mathscr{G}$, according to Definition 4.1.*

Proof. Let u be a solution to the minimum problem (4.1). Thanks to Lemma 4.2, we know that there exists $\Lambda > 0$ such that u is a solution to the minimization problem (4.4). Consider any competitor $v \in \mathcal{X}$, and let us prove the inequality (4.3). We can assume without loss of generality that $|\Omega_{h_v}| \leq |\Omega_{h_u}| + \mathscr{G}(u)$, since if the opposite inequality is in force then (4.3) holds trivially. We can therefore use the Lipschitz continuity of the nonlocal energy (see Proposition 2.2) with a uniform constant $L_{\mathcal{N}}$, depending ultimately only on u, together with the triangle inequality, and get

$$\begin{aligned}
\mathscr{G}(u) &\leq \mathscr{G}(v) + \gamma\big(\mathcal{N}(v) - \mathcal{N}(u)\big) + \Lambda\Big(\big||A_v| - m\big| - \big||A_u| - m\big| + \big||\Omega_{h_v}| - M\big| - \big||\Omega_{h_u}| - M\big|\Big) \\
&\leq \mathscr{G}(v) + \gamma L_{\mathcal{N}}|A_u \triangle A_v| + \gamma L_{\mathcal{N}}|B_u \triangle B_v| + \Lambda\Big(\big||A_v| - |A_u|\big| + \big||\Omega_{h_v}| - |\Omega_{h_u}|\big|\Big) \\
&\leq \mathscr{G}(v) + \gamma L_{\mathcal{N}}|A_u \triangle A_v| + \gamma L_{\mathcal{N}}|B_u \triangle B_v| + \Lambda\Big(\big|A_v \triangle A_u\big| + \big|\Omega_{h_v} \triangle \Omega_{h_u}\big|\Big) \\
&\leq \mathscr{G}(v) + \big(\gamma L_{\mathcal{N}} + 2\Lambda\big)\Big(\big|A_v \triangle A_u\big| + \big|B_v \triangle B_u\big|\Big),
\end{aligned}$$

where we used the inequality $\big||E| - |F|\big| \leq |E \triangle F|$ for measurable sets $E, F \subset \mathbb{R}^n$, and the inequality $|\Omega_{h_v} \triangle \Omega_{h_u}| \leq |A_v \triangle A_u| + |B_v \triangle B_u|$. □

Proposition 4.4 (Boundedness)**.** *Let $u \in \mathcal{A}_{\Lambda,M}$. Then $h_u \in L^\infty(Q_L)$.*

Proof. Along the proof, $C > 0$ will denote a constant depending only on the dimension and on the surface tension coefficients, that might change from line to line. Fix $u \in \mathcal{A}_{\Lambda,M}$. For each $t > 0$ define $E_t \coloneqq \Omega_{h_u} \cap \{x_n > t\}$ and $m(t) \coloneqq |E_t|$. Then for a.e. $t > 0$ it holds

$$m'(t) = -\mathcal{H}^{n-1}(\Omega_{h_u}^{(1)} \cap \{x_n = t\})$$

and $\mathcal{H}^{n-1}(\partial^* A_u \cap \{x_n = t\}) = \mathcal{H}^{n-1}(\partial^* B_u \cap \{x_n = t\}) = 0$. By quasi-minimality of u, comparing with the configuration $v \coloneqq u\chi_{\{x_n \leq t\}}$, we get

$$\begin{aligned}
\sigma_A \mathcal{H}^{n-1}(\Gamma_u^A \cap \{x_n > t\}) + \sigma_B \mathcal{H}^{n-1}(\Gamma_u^B \cap \{x_n > t\}) + \sigma_{AB}\mathcal{H}^{n-1}(\Gamma^{AB} \cap \{x_n > t\}) \\
\leq -\max\{\sigma_A, \sigma_B\} m'(t) + \Lambda m(t).
\end{aligned}$$

By using the isoperimetric inequality we can bound from below the left-hand side in the previous inequality as follows:

$$\begin{aligned}
-\max\{\sigma_A, \sigma_B\} m'(t) + \Lambda m(t) &\geq \min\{\sigma_A, \sigma_B\}\mathcal{H}^{n-1}\big((\Gamma_u^A \cup \Gamma_u^B) \cap \{x_n > t\}\big) \\
&= \min\{\sigma_A, \sigma_B\}\big(\mathscr{P}(E_t) + m'(t)\big) \\
&\geq C m(t)^{\frac{n-1}{n}} + C m'(t).
\end{aligned}$$

Now, given any $\varepsilon > 0$, by finiteness of the volume of Ω_{h_u} we can find $\bar{t} > 0$ such that $m(\bar{t}) < \varepsilon$. For $t > \bar{t}$, by writing $m(t) = m(t)^{\frac{1}{n}} m(t)^{\frac{n-1}{n}} \leq \varepsilon^{\frac{1}{n}} m(t)^{\frac{n-1}{n}}$, we therefore get

$$-\big(C + \max\{\sigma_A, \sigma_B\}\big) m'(t) \geq (C - \Lambda\varepsilon^{\frac{1}{n}}) m(t)^{\frac{n-1}{n}}.$$

By choosing $\varepsilon > 0$ small enough (depending ultimately on n, Λ, and on the surface tension coefficients) we obtain the differential inequality

$$m(t)^{\frac{n-1}{n}} \leq -Cm'(t) \qquad \text{for a.e. } t > \bar{t}.$$

Integrating this inequality from $\bar{t}$ to $t > \bar{t}$ we get $m(t)^{\frac{1}{n}} \leq \varepsilon^{\frac{1}{n}} - \frac{1}{C}(t-\bar{t})$. Thus, for $t > \bar{t} + C\varepsilon^{\frac{1}{n}}$, we get $m(t) = 0$. This concludes the proof. □

Remark 4.5. *In the proof of Proposition 4.4 we proved the following elimination-type property: there exists $\varepsilon > 0$ and $t_0 > 0$, depending on n, Λ and on the surface tension coefficients, such that if $u \in \mathcal{A}_{\Lambda,M}$ and*

$$|\Omega_{h_u} \cap \{x_n > t\}| < \varepsilon$$

for some $t > 0$, then

$$|\Omega_{h_u} \cap \{x_n > t + t_0\}| = 0.$$

4.2 Partial regularity of quasi-minimizers in dimension 2

From now on we assume that the dimension of the space is $n = 2$. We also assume that the surface tension coefficients satisfy the *strict* triangle inequalities

$$\sigma_{AB} < \sigma_A + \sigma_B, \quad \sigma_A < \sigma_B + \sigma_{AB}, \quad \sigma_B < \sigma_A + \sigma_{AB}. \tag{4.44}$$

Under these assumptions we will show a series of regularity properties satisfied by a quasi-minimizer u of the surface energy $\mathscr{G}$, according to Definition 4.1. Notice that for $u \in \mathcal{A}_{\Lambda,M}$ we have a uniform bound

$$\sup_{u \in \mathcal{A}_{\Lambda,M}} \Big(|\Omega_{h_u}| + \mathcal{H}^1(\Gamma_{h_u}) \Big) \leq C$$

for a constant C depending on M, Λ and on the surface tension coefficients. In two dimensions, this bound immediately yields boundedness from above of the film, that is there exists $\overline{M} > 0$ (depending on M, Λ and on the surface tension coefficients) such that

$$\overline{\Omega}_{h_u} \subset [0, L] \times [0, \overline{M}] \qquad \text{for all } u \in \mathcal{A}_{\Lambda,M}. \tag{4.45}$$

We first fix some notation to be used throughout this section. We let

$$\mathcal{Q} := \Big(-\frac{1}{2}, \frac{1}{2}\Big) \times \Big(-\frac{1}{2}, \frac{1}{2}\Big), \qquad \mathcal{Q}_r := r\mathcal{Q}, \qquad \mathcal{Q}_r(z_0) := z_0 + \mathcal{Q}_r,$$

for $z_0 \in \mathbb{R}^2$ and $r > 0$. We also define the strips

$$\mathcal{C}^- := \Big(-\frac{1}{2}, \frac{1}{2}\Big) \times \Big(-\infty, \frac{1}{2}\Big), \qquad \mathcal{C}_r^- := r\mathcal{C}^-, \qquad \mathcal{C}_r^-(z_0) := z_0 + \mathcal{C}_r^-.$$

The first regularity fact that we establish is an elimination property for the empty region above the film, in the spirit of [20].

Proposition 4.6 (Infiltration for V)**.** *Let $u \in \mathcal{A}_{\Lambda,M}$. There exists $\varepsilon_1 > 0$, depending on Λ, M, and on the surface tension coefficients, such that if for some $z_0 \in \mathbb{R}^2$ and $r \in (0, 1)$*

$$|V_u^\# \cap \mathcal{Q}_r(z_0)| < \varepsilon_1 r^2, \tag{4.46}$$

then

$$|V_u^\# \cap \mathcal{Q}_{\frac{r}{2}}(z_0)| = 0. \tag{4.47}$$

Proof. Along the proof, to lighten the notation we will drop the subscript u from the sets (2.11) of the partition determined by u and from the corresponding interfaces (2.13)–(2.14). The proof is divided into two steps.

Step 1: infiltration in strips. We first show that there exists $\tilde{\varepsilon}_1 > 0$, depending on Λ and on the surface tension coefficients, such that if for some $z_0 \in \mathbb{R}^2$ and $r \in (0,1)$

$$|V^{\#} \cap \mathcal{C}_r^-(z_0)| < \tilde{\varepsilon}_1 r^2, \tag{4.48}$$

then

$$|V^{\#} \cap \mathcal{C}_{\frac{3}{4}r}^-(z_0)| = 0. \tag{4.49}$$

We assume for notation convenience, and without loss of generality, that $\mathcal{C}_r^-(z_0) \subset (0,L) \times \mathbb{R}$; the general case is obtained by periodicity.

For $s \in [0,r]$ we let $m(s) := |V \cap \mathcal{C}_s^-(z_0)|$, so that $m(r) \le \tilde{\varepsilon}_1 r^2$ by the assumption (4.48). The function $m(s)$ is monotone nondecreasing, with

$$m'(s) = \frac{1}{2}\mathcal{H}^1(\partial\mathcal{C}_s^-(z_0) \cap V^{(1)}) \qquad \text{for } \mathscr{L}^1\text{-almost every } s > 0. \tag{4.50}$$

Fix now $s \in (0,r)$ such that (4.50) holds and $\mathcal{H}^1(J_u \cap \partial\mathcal{C}_s^-(z_0)) = 0$ (notice that $\mathscr{L}^1$-almost every $s > 0$ has this property). We define a competitor by "filling" the empty region in $\mathcal{C}_s^-(z_0)$ above the substrate by the phase A or B. More precisely, assume that

$$\mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0)) \le \mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) \tag{4.51}$$

and define (see Figure 6)

$$u_s(z) := \begin{cases} 1 & \text{if } z \in V \cap \mathcal{C}_s^-(z_0), \\ u(z) & \text{otherwise} \end{cases}$$

(which corresponds to fill the region $V \cap \mathcal{C}_s^-(z_0)$ by the phase A). The proof in the other case, when one has the opposite inequality in (4.51), follows similarly by filling $V \cap \mathcal{C}_s^-(z_0)$ by the phase B.

We then have $A_{u_s} = A \cup (V \cap \mathcal{C}_s^-(z_0))$, $B_{u_s} = B$, and $A_{u_s} \cup B_{u_s}$ is the subgraph of an admissible profile; therefore $u_s \in \mathcal{X}$ is an admissible configuration and by quasi-minimality of u we find

$$\begin{aligned} \mathscr{G}(u) &\le \mathscr{G}(u_s) + \Lambda\big(|A_{u_s} \triangle A| + |B_{u_s} \triangle B|\big) \\ &= \mathscr{G}(u) + \big(\mathscr{G}(u_s) - \mathscr{G}(u)\big) + \Lambda m(s) \\ &\le \mathscr{G}(u) - \sigma_A \mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) + (\sigma_{AB} - \sigma_B)\mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0)) \\ &\qquad + \sigma_A \mathcal{H}^1(V^{(1)} \cap \partial\mathcal{C}_s^-(z_0)) + (\sigma_{AS} - \sigma_S)\mathcal{H}^1(S^V \cap \mathcal{C}_s^-(z_0)) + \Lambda m(s). \end{aligned} \tag{4.52}$$

Observe now that by (4.51)

$$\begin{aligned} &-\sigma_A \mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) + (\sigma_{AB} - \sigma_B)\mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0)) \\ &\qquad \overset{(4.51)}{\le} -\sigma_A \mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) + \max\{\sigma_{AB} - \sigma_B, 0\}\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) \\ &\qquad = \max\{\sigma_{AB} - \sigma_B - \sigma_A, -\sigma_A\}\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) \end{aligned}$$

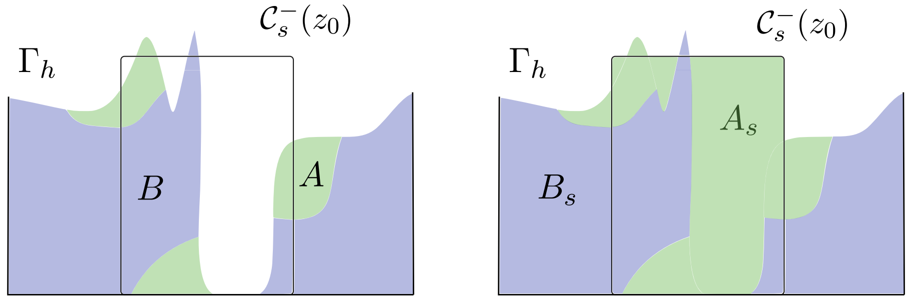

Figure 6: The construction of the competitor in Step 1 when (4.51) is in force: we fill the empty region in $\mathcal{C}_s^-(z_0)$ above the substrate by the phase A.

$$
\begin{aligned}
&= -2c_1\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) \\
&\overset{(4.51)}{\leq} -c_1\big(\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) + \mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0))\big)
\end{aligned}
$$

where we set $c_1 := -\frac{1}{2}\max\{\sigma_{AB}-\sigma_B-\sigma_A, -\sigma_A\} > 0$ thanks to the strict triangular inequality between the coefficients. Hence

$$
\begin{aligned}
-\sigma_A\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) &+ (\sigma_{AB}-\sigma_B)\mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0)) - \sigma_S\mathcal{H}^1(S^V \cap \mathcal{C}_s^-(z_0)) \\
&\leq -\min\{c_1,\sigma_S\}\big(\mathcal{H}^1(\Gamma^A \cap \mathcal{C}_s^-(z_0)) + \mathcal{H}^1(\Gamma^B \cap \mathcal{C}_s^-(z_0)) + \mathcal{H}^1(S^V \cap \mathcal{C}_s^-(z_0))\big) \\
&= -\min\{c_1,\sigma_S\}\mathscr{P}(V;\mathcal{C}_s^-(z_0)) \\
&= -\min\{c_1,\sigma_S\}\big(\mathscr{P}(V \cap \mathcal{C}_s^-(z_0)) - 2m'(s)\big) \\
&\leq -c_2|V \cap \mathcal{C}_s^-(z_0)|^{\frac{1}{2}} + c_3m'(s) \\
&= -c_2m(s)^{\frac{1}{2}} + c_3m'(s),
\end{aligned}
\tag{4.53}
$$

where we used the isoperimetric inequality in the last inequality, and c_2, c_3 are positive constants depending on the surface tension coefficients. Furthermore, by the geometry of the set V we have $\mathcal{H}^1(S^V \cap \mathcal{C}_s^-(z_0)) \leq \mathcal{H}^1(V^{(1)} \cap \partial\mathcal{C}_s^-(z_0))$, hence

$$
\begin{aligned}
\sigma_A\mathcal{H}^1(V^{(1)} \cap \partial\mathcal{C}_s^-(z_0)) + \sigma_{AS}\mathcal{H}^1(S^V \cap \mathcal{C}_s^-(z_0)) &\leq (\sigma_A+\sigma_{AS})\mathcal{H}^1(V^{(1)} \cap \partial\mathcal{C}_s^-(z_0)) \\
&= 2(\sigma_A+\sigma_{AS})m'(s).
\end{aligned}
\tag{4.54}
$$

By inserting (4.53)–(4.54) into (4.52) and setting $c_4 := c_3 + 2(\sigma_A+\sigma_{AS})$ we find

$$
\begin{aligned}
c_2m(s)^{\frac{1}{2}} &\leq c_4m'(s) + \Lambda m(s) \\
&\leq c_4m'(s) + \Lambda m(s)^{\frac{1}{2}}m(r)^{\frac{1}{2}} \\
&\leq c_4m'(s) + \sqrt{\tilde{\varepsilon}_1}\Lambda m(s)^{\frac{1}{2}}r.
\end{aligned}
$$

The previous estimate holds for almost every $s \in (0,r)$ obeying (4.51), but one can obtain the same estimate also for almost every s satisfying the opposite inequality (with possibly different constants c_2, c_4). Therefore

$$
\big(c_2 - \sqrt{\tilde{\varepsilon}_1}\Lambda\big)m(s)^{\frac{1}{2}} \leq c_4m'(s) \quad \text{for a.e. } s \in (0,r).
$$

Then, by choosing $\tilde{\varepsilon}_1 > 0$ small enough, depending on Λ and on the surface tension coefficients, we obtain that

$$m(s)^{\frac{1}{2}} \leq Cm'(s) \quad \text{for a.e. } s \in (0, r),$$

for a constant $C > 0$ depending only on the surface tension coefficients.

Suppose now by contradiction that $m(\frac{3}{4}r) > 0$. Then the previous estimate yields $\frac{\mathrm{d}}{\mathrm{d}s}(m(s)^{\frac{1}{2}}) \geq \frac{1}{2C}$ for almost every $s \in (\frac{3}{4}r, r)$, and by integrating in $(\frac{3}{4}r, r)$ we find

$$m\Big(\frac{3}{4}r\Big)^{\frac{1}{2}} \leq m(r)^{\frac{1}{2}} - \frac{r}{8C} \leq \Big(\sqrt{\tilde{\varepsilon}_1} - \frac{1}{8C}\Big) r < 0$$

by possibly taking a smaller $\tilde{\varepsilon}_1$, a contradiction. This shows that $m(\frac{3}{4}r) = 0$, that is (4.49).

Step 2. We now claim that there exists $\varepsilon_1 > 0$ such that

$$|V^{\#} \cap \mathcal{Q}_r(z_0)| < \varepsilon_1 r^2 \qquad \Longrightarrow \qquad |V^{\#} \cap \mathcal{C}^-_{\frac{3}{4}r}(z_0)| < \tilde{\varepsilon}_1 \Big(\frac{3}{4}r\Big)^2, \tag{4.55}$$

where $\tilde{\varepsilon}_1$ is given by the previous step. Once this claim is proved, the conclusion of the proposition follows easily by combining this property with Step 1.

Let us now prove (4.55). As before by periodicity we can assume $\mathcal{Q}_r(z_0) \subset (0, L) \times \mathbb{R}$. We denote by $\mathcal{Q}'_r(z_0) \coloneqq z_0 + (-\frac{r}{2}, \frac{r}{2}) \times \{-\frac{r}{2}\}$ the bottom side of the square $\mathcal{Q}_r(z_0)$. We first observe that, by the geometry of the set V, we have

$$\mathcal{H}^1(V^{(1)} \cap \mathcal{Q}'_r(z_0)) \leq \frac{|V \cap \mathcal{Q}_r(z_0)|}{r} \leq \varepsilon_1 r. \tag{4.56}$$

Then (assuming without loss of generality that $\varepsilon_1 < \frac{1}{8}$) we can find $\rho \in (\frac{3}{4}r, r)$ such that the two points $z_0 + (-\frac{\rho}{2}, -\frac{r}{2})$, $z_0 + (\frac{\rho}{2}, -\frac{r}{2})$, on $\mathcal{Q}'_r(z_0)$, are not points of density one for V. We then consider the strip

$$U \coloneqq \mathcal{C}^-_\rho(z_0) \backslash \overline{\mathcal{Q}_r(z_0)}$$

and, by the choice of ρ, it follows that the lateral boundary of U is outside $V^{(1)}$, hence

$$\mathcal{H}^1(V^{(1)} \cap \partial U) \leq \mathcal{H}^1(V^{(1)} \cap \mathcal{Q}'_r(z_0)) \overset{(4.56)}{\leq} \varepsilon_1 r. \tag{4.57}$$

We can further assume that

$$\mathcal{H}^1(J_u \cap \partial \mathcal{C}^-_\rho(z_0)) = 0, \tag{4.58}$$

since this is valid for $\mathscr{L}^1$-almost every $\rho \in (0, r)$.

To continue, we assume that

$$\mathcal{H}^1(\Gamma^A \cap U) \leq \mathcal{H}^1(\Gamma^B \cap U) \tag{4.59}$$

and we construct a competitor by filling the region $V \cap U$ by the phase B (the proof in the other case, when one has the opposite inequality in (4.59), follows similarly by filling $V \cap U$ by the phase A): we define

$$\tilde{u}(z) \coloneqq \begin{cases} -1 & \text{if } z \in V \cap U, \\ u(z) & \text{otherwise.} \end{cases}$$

Notice that $\tilde{u} \in \mathcal{X}$ is an admissible configuration, hence by quasi-minimality of u we find

$$\begin{aligned}
\mathscr{G}(u) &\leq \mathscr{G}(\tilde{u}) + \Lambda\big(|A_{\tilde{u}} \triangle A| + |B_{\tilde{u}} \triangle B|\big) \\
&= \mathscr{G}(u) + \big(\mathscr{G}(\tilde{u}) - \mathscr{G}(u)\big) + \Lambda|V \cap U| \\
&\leq \mathscr{G}(u) + (\sigma_{AB} - \sigma_A)\mathcal{H}^1(\Gamma^A \cap U) - \sigma_B \mathcal{H}^1(\Gamma^B \cap U) \\
&\qquad + \sigma_B \mathcal{H}^1(V^{(1)} \cap \partial U) + (\sigma_{BS} - \sigma_S)\mathcal{H}^1(S^V \cap U) + \Lambda|V \cap U|.
\end{aligned} \tag{4.60}$$

Now, arguing similarly to the proof of (4.53), using the assumption (4.59), we find

$$\begin{aligned}
(\sigma_{AB} - \sigma_A)&\mathcal{H}^1(\Gamma^A \cap U) - \sigma_B \mathcal{H}^1(\Gamma^B \cap U) - \sigma_S \mathcal{H}^1(S^V \cap U) \\
&\leq \max\{\sigma_{AB} - \sigma_A - \sigma_B, -\sigma_B\}\mathcal{H}^1(\Gamma^B \cap U) - \sigma_S \mathcal{H}^1(S^V \cap U) \\
&\leq -c_1\big(\mathcal{H}^1(\Gamma^B \cap U) + \mathcal{H}^1(\Gamma^A \cap U) + \mathcal{H}^1(S^V \cap U)\big) \\
&= -c_1 \mathscr{P}(V; U) \\
&= -c_1\big(\mathscr{P}(V \cap U) - \mathcal{H}^1(V^{(1)} \cap \partial U)\big) \\
&\leq -c_2|V \cap U|^{\frac{1}{2}} + c_1 \mathcal{H}^1(V^{(1)} \cap \partial U),
\end{aligned}$$

where c_1, c_2 are strictly positive constants depending on the surface tension coefficients, and we used the isoperimetric inequality in the last passage. By inserting this inequality into (4.60) we obtain

$$c_2|V \cap U|^{\frac{1}{2}} \leq (c_1 + \sigma_B)\mathcal{H}^1(V^{(1)} \cap \partial U) + \sigma_{BS}\mathcal{H}^1(S^V \cap U) + \Lambda|V \cap U|.$$

Now observe that, by the geometry of the set V, we have $\mathcal{H}^1(S^V \cap U) \leq \mathcal{H}^1(V^{(1)} \cap \mathcal{Q}'_r(z_0))$; hence from the previous inequality it follows that

$$\begin{aligned}
c_2|V \cap U|^{\frac{1}{2}} &\leq (c_1 + \sigma_B)\mathcal{H}^1(V^{(1)} \cap \partial U) + \sigma_{BS}\mathcal{H}^1(V^{(1)} \cap \mathcal{Q}'_r(z_0)) + \Lambda|V \cap U| \\
&\overset{(4.57)}{\leq} (c_1 + \sigma_B + \sigma_{BS})\varepsilon_1 r + \Lambda|V \cap U|.
\end{aligned}$$

Finally, observe that by using the uniform bound (4.45) and the vertical geometry of V we have $|V \cap U| \leq \overline{M}\mathcal{H}^1(V^{(1)} \cap \mathcal{Q}'_r(z_0)) \leq \varepsilon_1 \overline{M} r$, so that by inserting this estimate in the previous inequality we obtain

$$c_2|V \cap U|^{\frac{1}{2}} \leq (c_1 + \sigma_B + \sigma_{BS})\varepsilon_1 r + \varepsilon_1 \Lambda \overline{M} r,$$

that is,

$$|V \cap U| \leq C\varepsilon_1^2 r^2$$

for some constant $C > 0$ depending on Λ, M and on the surface tension coefficients (recall that the bound $\overline{M}$ in (4.45) depends only on these quantities). Eventually

$$|V \cap \mathcal{C}^-_{\frac{3}{4}r}(z_0)| \leq |V \cap U| + |V \cap \mathcal{Q}_r(z_0)| \leq C\varepsilon_1^2 r^2 + \varepsilon_1 r^2 \leq \tilde{\varepsilon}_1\Big(\frac{3}{4}r\Big)^2,$$

provided that we choose ε_1 small enough. This completes the proof of (4.55). □

We also have a dual statement for the region occupied by the film. The proof follows by the same argument used in the proof of Proposition 4.6. However, since we can obtain this result as a consequence of the stronger property proved in Proposition 4.9, we omit the proof.

Proposition 4.7 (Infiltration for $A \cup B$)**.** *Let $u \in \mathcal{A}_{\Lambda,M}$. There exists $\varepsilon_2 > 0$, depending on Λ, M, and on the surface tension coefficients, such that if for some $z_0 \in \mathbb{R}^2$ and $r \in (0,1)$ such that $\mathcal{Q}_r(z_0) \cap S = \emptyset$ we have*

$$|\Omega^{\#}_{h_u} \cap \mathcal{Q}_r(z_0)| < \varepsilon_2 r^2, \tag{4.61}$$

then

$$|\Omega^{\#}_{h_u} \cap \mathcal{Q}_{\frac{r}{2}}(z_0)| = 0. \tag{4.62}$$

Remark 4.8. *It is worth to notice that, in the proof of Proposition 4.6 (and also of Proposition 4.7), the constraint of being subgraphs prevents us to construct competitors by means of local variations in a square $\mathcal{Q}_r(z_0)$, but imposes to consider the full vertical region below or above the square. Notice also that, if a single set Ω_h is a quasi-minimizer of the perimeter in the class of subgraphs, a standard argument [16, Theorem 14.8] shows that it is actually a quasi-minimizer among* all *possible competitors of finite perimeter, without the constraint (we will exploit this fact in the proof of Proposition 4.13). However, in our case we have a partition of the subgraph into two sets A, B of finite perimeter, and this argument fails due to the presence of different types of interfaces between the phases.*

We continue by proving that a quasi-minimizer satisfies an interior ball condition. The proof of this result follows a strategy devised in [8] (see also [14, 15]) and adapted to our setting. Since we are in dimension 2, the function h_u^- (see (2.1)) is a lower semicontinuous representative of h_u; in the following it will be convenient to identify h_u with h_u^-, so that in particular the subgraph $\Omega^{\#}_{h_u}$ is an open set. From now on, we work under this convention.

Proposition 4.9 (Interior ball)**.** *Let $u \in \mathcal{A}_{\Lambda,M}$ and let $\rho_0 < \frac{\min\{\sigma_A,\sigma_B\}}{\Lambda}$. Then for every $\bar{z} \in \Gamma^{\#}_{h_u}$ there exists an open ball $B_{\rho_0}(z_0)$ such that*

$$B_{\rho_0}(z_0) \subset \big\{(x,y) \in \mathbb{R}^2 \,:\, y < h_u(x)\big\}, \qquad \partial B_{\rho_0}(z_0) \cap \Gamma^{\#}_{h_u} = \{\bar{z}\}. \tag{4.63}$$

Proof. We divide the proof into two steps. To simplify the notation we drop the subscripts on the various objects depending on u, which is fixed along this proof. We also denote by

$$\Omega_h^- \coloneqq \big\{(x,y) \in \mathbb{R}^2 \,:\, y < h(x)\big\} \tag{4.64}$$

where $h = h_u$ is the profile associated with the configuration u. Recall that, by the convention of identifying h with its lower semicontinuous representative h^-, the set Ω_h^- is open.

Step 1. We claim that for every ball $B_{\rho_0}(z_0) \subset \Omega_h^-$, where ρ_0 is as in the statement, the set $\partial B_{\rho_0}(z_0) \cap \Gamma^{\#}_h$ consists of at most one point.

Suppose on the contrary that there exists a ball $B_\rho(z_0) \subset \Omega_h^-$, $z_0 = (x_0, y_0)$, such that $\partial B_\rho(z_0) \cap \Gamma^{\#}_h$ contains at least two points $a = (x_a, y_a)$, $b = (x_b, y_b)$, with $x_a \le x_b$, $h^-(x_a) \le y_a \le h^+(x_a)$, $h^-(x_b) \le y_b \le h^+(x_b)$. We will prove that this is not possible if $\rho \le \rho_0$. By periodicity, we assume without loss of generality that $B_\rho(z_0) \subset (0,L) \times \mathbb{R}$.

We define (see Figure 7) $\Gamma_{a,b}$ to be the arc on Γ_h connecting a with b, and $\gamma_{a,b}$ to be the arc on $\partial B_\rho(z_0) \cap \{y \ge y_0\}$ connecting a with b. Notice that by construction $\Gamma_{a,b}$ lies above $\gamma_{a,b}$. We also let D to be the region enclosed by $\Gamma_{a,b}$ and $\gamma_{a,b}$ (i.e. the bounded component of $\mathbb{R}^2 \backslash (\Gamma_{a,b} \cup \gamma_{a,b})$). Also let

$$L_A \coloneqq \mathcal{H}^1(\Gamma_{a,b} \cap \Gamma^A), \quad L_B \coloneqq \mathcal{H}^1(\Gamma_{a,b} \cap \Gamma^B), \quad L \coloneqq L_A + L_B = \mathcal{H}^1(\Gamma_{a,b}),$$

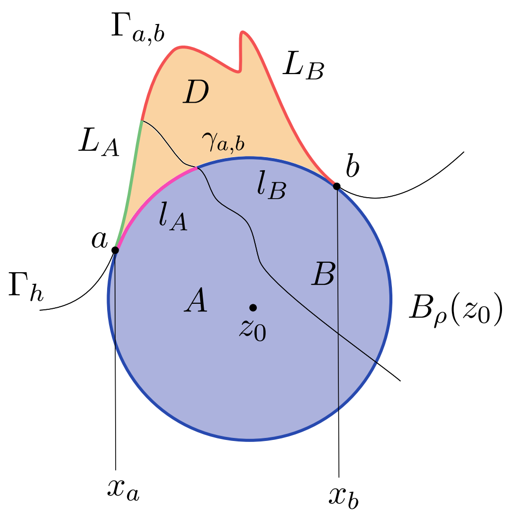

Figure 7: The construction in the proof of Proposition 4.9.

$$\ell_A := \mathcal{H}^1\big(\gamma_{a,b} \cap \partial^*(A \cap B_\rho(z_0))\big), \quad \ell_B := \mathcal{H}^1(\gamma_{a,b}) - \ell_A, \quad \ell := \ell_A + \ell_B = \mathcal{H}^1(\gamma_{a,b}).$$

We construct a competitor by removing the set D from the subgraph Ω_h: more precisely, we define $\tilde{u} := u\chi_{D^c}$. In this way $\tilde{u}$ is an admissible configuration, $A_{\tilde{u}} = A \backslash D$, $B_{\tilde{u}} = B \backslash D$, and the profile $h_{\tilde{u}}$ coincides with h outside $[x_a, x_b]$, and its graph on $[x_a, x_b]$ is given by $\gamma_{a,b}$. Then, by using the quasi-minimality of u we obtain

$$\begin{aligned} \mathscr{G}(u) &\leq \mathscr{G}(\tilde{u}) + \Lambda|D| \\ &= \mathscr{G}(u) - \sigma_A L_A - \sigma_B L_B - \sigma_{AB}\mathcal{H}^1(\Gamma^{AB} \cap \overline{D}) + \sigma_A \ell_A + \sigma_B \ell_B + \Lambda|D|, \end{aligned}$$

hence

$$\sigma_A(L_A - \ell_A) + \sigma_B(L_B - \ell_B) + \sigma_{AB}\mathcal{H}^1(\Gamma^{AB} \cap \overline{D}) \leq \Lambda|D|. \tag{4.65}$$

To continue, we distinguish between two cases.

Case 1: assume that $\ell_A \leq L_A$, $\ell_B \leq L_B$. In this case we obtain from (4.65)

$$\min\{\sigma_A, \sigma_B\}(L - \ell) \leq \Lambda|D|. \tag{4.66}$$

We estimate the difference $L - \ell$ as in [15, Lemma 6.6]; we reproduce the argument here for the reader's convenience. Assume first that h is Lipschitz continuous. Since $\Gamma_{a,b}$ and $\gamma_{a,b}$ are the graphs on $[x_a, x_b]$ of h and $h_{\tilde{u}}$, respectively, and $h(x_a) = h_{\tilde{u}}(x_a)$, $h(x_b) = h_{\tilde{u}}(x_b)$, we have

$$\begin{aligned} L - \ell = \mathcal{H}^1(\Gamma_{a,b}) - \mathcal{H}^1(\gamma_{a,b}) &= \int_{x_a}^{x_b} \Big(\sqrt{1 + (h'(x))^2} - \sqrt{1 + (h'_{\tilde{u}}(x))^2}\Big)\,\mathrm{d}x \\ &\geq \int_{x_a}^{x_b} \frac{h'_{\tilde{u}}(x)}{\sqrt{1 + (h'_{\tilde{u}}(x))^2}}\big(h'(x) - h'_{\tilde{u}}(x)\big)\,\mathrm{d}x \\ &= -\int_{x_a}^{x_b} \bigg(\frac{h'_{\tilde{u}}(x)}{\sqrt{1 + (h'_{\tilde{u}}(x))^2}}\bigg)'\big(h(x) - h_{\tilde{u}}(x)\big)\,\mathrm{d}x \\ &= \frac{1}{\rho}\int_{x_a}^{x_b} \big(h(x) - h_{\tilde{u}}(x)\big)\,\mathrm{d}x = \frac{1}{\rho}|D|. \end{aligned}$$

By combining this inequality with (4.66), we see that necessarily $\rho \geq \frac{\min\{\sigma_A,\sigma_B\}}{\Lambda}$. Therefore, any ball $B_{\rho_0}(z_0) \subset \Omega_h^-$ can touch the graph $\Gamma_h^\#$ at most once.

If h is not Lipschitz, then we can approximate h by a sequence of Lipschitz functions g_k as in [15, Lemma 6.2], write the previous inequality for g_k and obtain the same conclusion by passing to the limit.

Case 2: assume that $\ell_A > L_A$, $\ell_B \leq L_B$ (the other case, $\ell_A \leq L_A$ and $\ell_B > L_B$, is completely analogous). In this case, thanks to the triangle inequality $\sigma_A < \sigma_B + \sigma_{AB}$ (see (4.44)), we find by (4.65)

$$(\sigma_B + \sigma_{AB})(L_A - \ell_A) + \sigma_B(L_B - \ell_B) + \sigma_{AB}\mathcal{H}^1(\Gamma^{AB} \cap \overline{D}) \leq \Lambda|D|,$$

and in turn

$$\sigma_B(L - \ell) + \sigma_{AB}(L_A - \ell_A + \mathcal{H}^1(\Gamma^{AB} \cap \overline{D})) \leq \Lambda|D|.$$

One can check that

$$L_A - \ell_A + \mathcal{H}^1(\Gamma^{AB} \cap \overline{D}) \geq 0 \tag{4.67}$$

(this inequality follows essentially by [22, Ex. 15.14]). Therefore, using (4.67) it follows that $\sigma_B(L-\ell) \leq \Lambda|D|$ and in particular (4.66) holds. We can therefore repeat the same argument as in the previous case.

Step 2. We now prove the existence of an interior ball at every point of $\Gamma_h^\#$, by the same argument as in [8, Lemma 2] or [14, Proposition 3.3, Step 2]. Let U be the union of all balls of radius ρ_0 which are contained in Ω_h^-. We claim that

$$\Omega_h^- \subset U, \tag{4.68}$$

a fact which implies the conclusion of the proposition. If (4.68) does not hold, then by connectedness of Ω_h^- there exists $z_0 \in \Omega_h^- \cap \partial U$. We consider a sequence $z_n \to z_0$, $z_n \in U$, and for every n we can choose a ball $B_{\rho_0}(w_n) \subset \Omega_h^-$ containing z_n. Up to subsequences, the balls $B_{\rho_0}(w_n)$ converge in the Hausdorff metric to a limit ball $B_{\rho_0}(w_0) \subset \Omega_h^-$ such that $z_0 \in \partial B_{\rho_0}(w_0)$.

By the previous step, $B_{\rho_0}(w_0)$ intersects $\Gamma_h^\#$ no more than once; on the other hand the intersection must be non empty, or else we could slightly translate the ball remaining inside Ω_h^-, so that z_0 would be a point in U, contradicting the fact that $z_0 \in \partial U$.

Therefore $\partial B_{\rho_0}(w_0) \cap \Gamma_h^\# = \{\bar{z}\}$. If $(\bar{z} - w_0)\cdot(z_0 - w_0) < 0$, then we could slightly translate the ball as before and obtain that $z_0 \in U$, a contradiction. If else $(\bar{z} - w_0)\cdot(z_0 - w_0) \geq 0$, then we could rotate the ball around z_0, slightly away from $\bar{z}$, to obtain a new ball B of radius ρ_0, with $\overline{B} \subset \Omega_h^-$ and $z_0 \in \partial B$; by translating this ball towards z_0, we obtain that z_0 is in the interior of a ball of radius ρ_0 contained in Ω_h^-, which again implies $z_0 \in U$, a contradiction.

This completes the proof of (4.68). □

As a consequence of the interior ball condition proved in Proposition 4.9, we obtain the Lipschitz regularity of the free profile $\Gamma_{h_u}^\#$ of a quasi-minimizer u outside a finite set.

Proposition 4.10 (Lipschitz regularity)**.** *Let $u \in \mathcal{A}_{\Lambda,M}$. There exists a finite set $\Sigma \subset Q_L$, with $J_{h_u} \subset \Sigma$, such that h_u is locally Lipschitz in $Q_L \backslash \Sigma$ and has left and right derivatives at every point of $Q_L \backslash \Sigma$, that are respectively left and right continuous.*

Proof. We denote by $h := h_u$ the admissible profile associated with the configuration u. For $z \in \Gamma_h^\#$, we let

$$n(z) := \{\nu \in \mathbb{S}^1 \,:\, B_{\rho_0}(z + \rho_0 \nu) \subset \Omega_h^-\}$$

where $\rho_0 > 0$ is given by Proposition 4.9 and Ω_h^- is defined in (4.64). In view of Proposition 4.9, $n(z) \neq \emptyset$ for all $z \in \Gamma_h^\#$. Notice also that, if $\nu \in n(z)$ for some z, then necessarily $\nu \cdot e_2 \leq 0$ (or otherwise $\Gamma_h^\#$ would not be an extended graph). We define the singular set

$$\Sigma := \pi_x\big(\{z \in \Gamma_h \,:\, e_1 \in n(z) \text{ or } -e_1 \in n(z)\}\big) \subset Q_L, \tag{4.69}$$

where π_x denotes the projection on the x-axis, and by $\Sigma^\#$ its periodic extension. The plan of the proof is the following: we first identify some properties of the set $n(z)$, for $z \in \Gamma_h^\#$, and then we prove that Σ is a finite set, $J_h \subset \Sigma$, and h is locally Lipschitz outside Σ.

Step 1: $n(z)$ cannot contain both vectors e_1, $-e_1$.

This is a consequence of the infiltration property in Proposition 4.6. Indeed, on the contrary we would have $B_{\rho_0}(z + \rho_0 e_1) \cup B_{\rho_0}(z - \rho_0 e_1) \subset \Omega_h^-$ and z would be a cusp point. The Lebesgue two-dimensional density of the void $V_u^\#$ at such point is zero:

$$\lim_{r \to 0} \frac{|V_u^\# \cap \mathcal{Q}_r(z)|}{|\mathcal{Q}_r(z)|} = 0.$$

Hence the condition (4.46) is satisfied at small scale, and $|V_u^\# \cap \mathcal{Q}_r(z)| = 0$ for $r > 0$ small enough by Proposition 4.6 . This, however, is not possible since $z \in \Gamma_h^\#$.

Step 2: $n(z)$ is a (possibly degenerate) arc on $\mathbb{S}^1$ with length strictly smaller than π.

In view of the previous step and recalling that $\nu \cdot e_2 \leq 0$ for all $\nu \in n(z)$, it is enough to show that if $\nu_1, \nu_2 \in n(z)$ then, denoting by $[\nu_1, \nu_2]$ the shortest arc on $\mathbb{S}^1$ connecting ν_1 and ν_2, we have $[\nu_1, \nu_2] \subset n(z)$. Indeed, if $\nu \in [\nu_1, \nu_2]$ there exists $\rho > 0$ such that

$$B_\rho(z + \rho\nu) \subset B_{\rho_0}(z + \rho_0 \nu_1) \cup B_{\rho_0}(z + \rho_0 \nu_2) \subset \Omega_h^-.$$

It follows that also $B_{\rho_0}(z + \rho_0 \nu) \subset \Omega_h^-$, or otherwise there would be $\rho' \in [\rho, \rho_0]$ such that $\partial B_{\rho'}(z + \rho'\nu)$ meets $\Gamma_h^\#$ twice, which is excluded by Proposition 4.9 (see in particular Step 1 in the proof). Hence $\nu \in n(z)$.

Step 3: if $z_n, z_0 \in \Gamma_h^\#$, $z_n \to z_0$, and $\nu_n \in n(z_n)$, $\nu_n \to \nu_0$, then $\nu_0 \in n(z_0)$. In particular, $n(z)$ is closed for every $z \in \Gamma_h^\#$.

Indeed, the balls $\overline{B_{\rho_0}(z_n + \rho_0 \nu_n)} \subset \overline{\Omega_h^-}$ converge to $\overline{B_{\rho_0}(z_0 + \rho_0 \nu)}$ in the Hausdorff distance, hence this ball is also contained in $\overline{\Omega_h^-}$. It follows that $\nu_0 \in n(z_0)$.

Step 4: Σ is a finite set and $J_h \subset \Sigma$.

The finiteness of Σ follows by a compactness argument. Assume by contradiction that there are infinitely many points $x_n \in \Sigma$. Let $z_n = (x_n, y_n) \in \Gamma_h$ be corresponding points on the graph and assume, without loss of generality, that $e_1 \in n(z_n)$ for all n. Then up to subsequences $z_n \to z = (x, y) \in \Gamma_h$ and by Step 3 $e_1 \in n(z)$, so that $x \in \Sigma$. However, since the balls $B_{\rho_0}(z_n + \rho_0 e_1)$ are tangent to Γ_h at z_n and contained in the subgraph Ω_h^-, this configuration is possible only if the points z_n are vertically aligned for all n sufficiently large, which contradicts the assumption that the points x_n are distinct.

If $x \in J_h$, then $\Gamma_h \cap (\{x\} \times \mathbb{R})$ is a vertical segment $\{x\} \times [h^-(x), h^+(x)]$. At each point $z = (x, y)$ with $h^-(x) < y < h^+(x)$ it must be $n(z) = \{e_1\}$ or $n(z) = \{-e_1\}$, hence $x \in \Sigma$.

Step 5: h is locally Lipschitz in $\mathbb{R}\backslash\Sigma^\#$.

Let $z_0 = (x_0, y_0) \in \Gamma_h^\#$ with $x_0 \notin \Sigma^\#$. We can find $\delta > 0$ such that $\nu \cdot e_2 \leq -2\delta$ for all $\nu \in n(z_0)$. Thanks to Step 3, there exists $r_0 > 0$ such that

$$\nu \cdot e_2 \leq -\delta \qquad \text{for all } \nu \in n(z), \text{ for all } z \in \Gamma_h^\# \cap \big(z_0 + (-r_0, r_0)^2\big).$$

Let now $\bar{z} = (\bar{x}, \bar{y}) \in \Gamma_h^\# \cap (z_0 + (-r_0, r_0)^2)$ and let $\nu = (\nu_1, \nu_2) \in n(\bar{z})$. Notice that $\bar{x} \notin \Sigma^\#$ and therefore $\bar{x}$ is not a jump point of h, so that $\bar{y} = h(\bar{x})$. By definition the sphere $\partial B_{\rho_0}(\bar{z} + \rho_0 \nu)$ is tangent to $\Gamma_h^\#$ at the point $\bar{z}$, and in a neighbourhood of $\bar{z}$ it is the graph of the function

$$\phi(x) := \bar{y} + \rho_0 \nu_2 + \rho_0 |\nu_2| \sqrt{1 + \frac{2\nu_1}{\rho_0 \nu_2^2}(x - \bar{x}) - \frac{(x - \bar{x})^2}{\rho_0^2 \nu_2^2}}\,.$$

By construction, $h(\bar{x}) = \phi(\bar{x})$ and $h(x) \geq \phi(x)$ for all x in a neighoburhood of $\bar{x}$, since the ball $B_{\rho_0}(\bar{z} + \rho_0 \nu)$ is contained in Ω_h^-. By straightforward computations

$$\phi'(\bar{x}) = \frac{\nu_1}{|\nu_2|}\,, \qquad \phi''(\bar{x}) = \frac{1}{\rho_0 \nu_2^3} \geq -\frac{1}{\rho_0 \delta^3}\,.$$

It follows that the function $\phi(x) + \frac{x^2}{2\rho_0\delta^3}$ is smooth and convex in a neighbourhood of $\bar{x}$: therefore, there exists $\bar{r} > 0$ (depending on $\bar{x}$) such that for all $x \in (\bar{x} - \bar{r}, \bar{x} + \bar{r})$

$$\phi(x) + \frac{x^2}{2\rho_0\delta^3} \geq \phi(\bar{x}) + \frac{\bar{x}^2}{2\rho_0\delta^3} + \left(\frac{\nu_1}{|\nu_2|} + \frac{\bar{x}}{\rho_0\delta^3}\right)(x - \bar{x}).$$

In view of the relation between h and ϕ, the same inequality holds by replacing ϕ by h.

In conclusion, we proved the following property: given any $\bar{x} \in (x_0 - r_0, x_0 + r_0)$, there exists $\bar{r} > 0$ such that

$$h(x) + \frac{x^2}{2\rho_0\delta^3} \geq h(\bar{x}) + \frac{\bar{x}}{2\rho_0\delta^3} + \left(\frac{\nu_1}{|\nu_2|} + \frac{\bar{x}}{\rho_0\delta^3}\right)(x - \bar{x}) \quad \text{for all } x \in (\bar{x} - \bar{r}, \bar{x} + \bar{r})$$

(where $\nu = (\nu_1, \nu_2) \in n(\bar{x}, h(\bar{x}))$). This shows that the function $h(x) + \frac{x^2}{2\rho_0\delta^3}$ is convex in $(x_0 - r_0, x_0 + r_0)$. $\square$

Remark 4.11. *The points in the singular set Σ identified in Proposition 4.10 are of two possible kinds: they are either jump points of the function h_u, or continuity points of h_u at which the left or the right derivative of h_u is infinite. At the upper point $(x, h_u^+(x))$ of a jump $x \in J_{h_u}$, the graph has a vertical tangent. Notice also that the graph of h_u does not contain cusp points as a consequence of the infiltration property and the inner ball condition (see Step 1 in the proof of Proposition 4.10).*

Remark 4.12. *Let $u \in \mathcal{A}_{\Lambda,M}$. Since the set $\Omega_h^\# = \{(x,y) \in \mathbb{R}^2 : 0 < y < h(x)\}$ is open, A is a quasi-minimizer of the perimeter in $\Omega_h^\#$ in the classical sense. Thus it is possible to apply standard regularity results (see [22, Theorems 26.5 and 28.1]) to obtain that Γ_u^{AB} is a locally a $C^{1,\alpha}$-curve in $\Omega_h^\#$ for every $\alpha \in (0, 1/2)$, and that it coincides with $\partial A \cap \partial B$ in $\Omega_h^\#$.*

Finally, we show that the graph as a better regularity around points of $\partial^* A \cup \partial^* B$. Notice that if $h_u(x_0) = 0$, then $(x_0, h_u(x_0)) \notin \partial^* A \cup \partial^* B$, or otherwise V_u would have Lebesgue density zero at that point, which is not permitted by the infiltration property in Proposition 4.6.

Proposition 4.13. *Let $u \in \mathcal{A}_{\Lambda,M}$. If $x_0 \in Q_L \setminus \Sigma$ is such that $(x_0, h_u(x_0)) \in \partial^* A \cup \partial^* B$, then h_u is of class $C^{1,\alpha}$ in a neighbourhood of x_0, for every $\alpha \in (0, 1/2)$.*

Proof. To simplify the notation we denote by $h := h_u$ the admissible profile associated with the configuration u, and we remove the subscript u from the sets of the corresponding partition. Recalling (4.44), we let

$$\delta := \min\{\sigma_{AB} + \sigma_A - \sigma_B, \sigma_{AB}, \sigma_A\} > 0. \tag{4.70}$$

Step 1. Fix x_0 as in the statement and let $z_0 := (x_0, h(x_0))$. Thanks to Proposition 4.10 we can find $r_0 > 0$ (depending on x_0) such that h is Lipschitz continuous in $(x_0 - r_0, x_0 + r_0)$, with Lipschitz constant $\ell \in (0, \infty)$. For $s > 0$, set

$$R_s := z_0 + sR, \qquad R := (-1, 1) \times (-2\ell, 2\ell).$$

Then

$$\Gamma_h \cap \partial^{\pm} R_s = \emptyset \tag{4.71}$$

for all $s \in (0, r_0)$, where

$$\partial^{\pm} R_s := z_0 + (-s, s) \times \{\pm 2\ell s\}.$$

Moreover, by possibly reducing the value of r_0, we can also assume that $R_s \cap S = \emptyset$.

We now prove an infiltration-type property, similar to Proposition 4.6, for the two phases A, B at the point z_0. Precisely, we claim that there exists $\varepsilon > 0$ (depending on z_0) such that if

$$|A \cap R_r| \le \varepsilon r^2, \tag{4.72}$$

for some $0 < r < r_0$, then

$$|A \cap R_{r/2}| = 0. \tag{4.73}$$

The same property holds if the set A is replaced by the set B.

For $s \in (0, r_0)$ set $m(s) := |A \cap R_s|$. Then for $\mathscr{L}^1$-a.e. $s \in (0, r_0)$ we have that

$$m'(s) = 2\ell\, \mathcal{H}^1(A^{(1)} \cap (\partial^+ R_s \cup \partial^- R_s)) + \mathcal{H}^1(A^{(1)} \cap \partial R_s \setminus (\partial^+ R_s \cup \partial^- R_s)) \tag{4.74}$$

and that

$$\mathcal{H}^1(\partial R_s \cap (\partial^* A \cup \partial^* B)) = 0. \tag{4.75}$$

We claim that there exist $C_1, C_2 > 0$ such that for $s \in (0, r_0)$ satisfying (4.74) and (4.75), the following differential inequality is true:

$$C_1 m(s)^{\frac{1}{2}} \le C_2 m'(s) + 3\Lambda m(s). \tag{4.76}$$

Once (4.76) is established, (4.73) will follow by a standard argument by using (4.72) and choosing ε sufficiently small, as in the last part of Step 1 in the proof of Proposition 4.6.

We are thus left with proving (4.76). The idea is to construct a suitable competitor and to use the quasi-minimality inequality for u; we have to pay attention that the competitor satisfies the graph constraint. If

$$\mathcal{H}^1(\Gamma^A \cap R_s) > \mathcal{H}^1(\Gamma^{AB} \cap R_s) \tag{4.77}$$

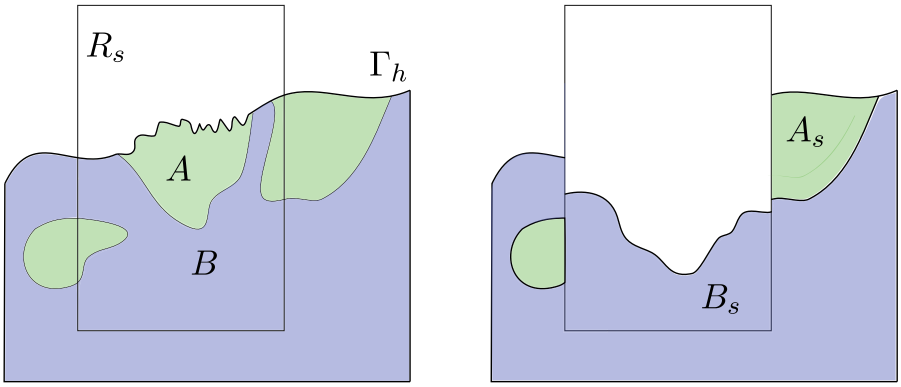

Figure 8: The construction of the competitor in the case $\mathcal{H}^1(\Gamma^A \cap R_s) > \mathcal{H}^1(\Gamma^{AB} \cap R_s)$: we remove $A \cap R_s$ and we 'move down' $B \cap R_s$.

then we set (see Figure 8)

$$A_s := A \backslash R_s, \qquad B_s := (B \backslash R_s) \cup (B \cap R_s)^G, \tag{4.78}$$

where, for a measurable set $E \subset R_s$ we define

$$E^G := \big\{(x, y) \in R_s \,:\, h(x_0) - 2\ell s \leq y \leq h(x_0) - 2\ell s + \mathcal{H}^1(E_x)\big\}, \tag{4.79}$$

with $E_x := \{t \in \mathbb{R} \,:\, (x, t) \in E\}$. In the case where

$$\mathcal{H}^1(\Gamma^A \cap R_s) \leq \mathcal{H}^1(\Gamma^{AB} \cap R_s) \tag{4.80}$$

we set instead (see Figure 9)

$$A_s := A \backslash R_s, \qquad B_s := B \cup (A \cap R_s)\,. \tag{4.81}$$

Note that the configuration $v_s = \chi_{A_s} - \chi_{B_s}$ is an admissible competitor for the quasi-minimality inequality in Definition 4.1. In the first case this follows from (4.71), while in the second case the free profile of the configuration is left unchanged. Denote by $h_s : Q_L \to [0, \infty)$ the admissible profile such that $\Omega_{h_s} = A_s \cup B_s$.

Assume that (4.77) holds, and thus A_s and B_s are defined as in (4.78). Then, by an argument similar to [16, Lemma 14.7] one can prove that

$$\mathcal{H}^1\left(\partial^* B_s \cap R_s\right) \leq \mathcal{H}^1(\partial^* B \cap R_s) + \mathcal{H}^1(A^{(1)} \cap \partial^- R_s). \tag{4.82}$$

Moreover, one can check that

$$|h_s^+(x_0 - s) - h_s^-(x_0 - s)| + |h_s^+(x_0 + s) - h_s^-(x_0 + s)| \leq \mathcal{H}^1(A^{(1)} \cap \partial R_s) \overset{(4.74)}{\leq} m'(s), \tag{4.83}$$

and that

$$|B_s \triangle B| \leq 2|A \cap R_s| = 2m(s). \tag{4.84}$$

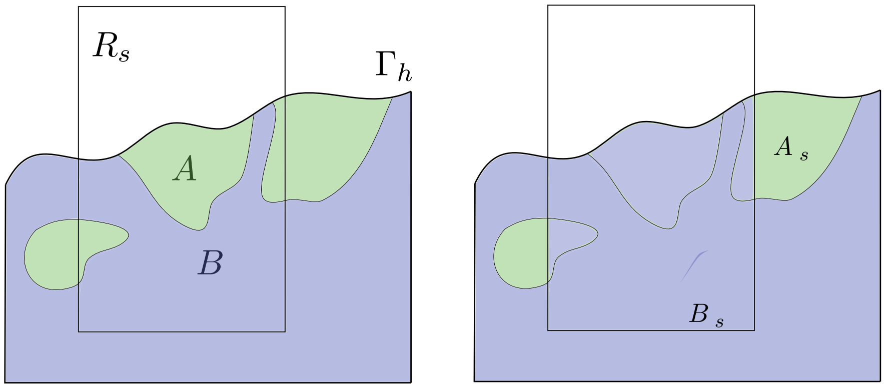

Figure 9: The construction of the competitor in the case $\mathcal{H}^1(\Gamma^A \cap R_s) \leq \mathcal{H}^1(\Gamma^{AB} \cap R_s)$: we simply fill $A \cap R_s$ with B.

The quasi-minimality of u, together with (4.74), (4.75), (4.82), (4.83), and (4.84), yields

$$\begin{aligned}
&\sigma_A \mathcal{H}^1(\Gamma^A \cap R_s) + \sigma_B \mathcal{H}^1(\Gamma^B \cap R_s) + \sigma_{AB} \mathcal{H}^1(\Gamma^{AB} \cap R_s) \\
&\qquad \leq \sigma_B \mathcal{H}^1(\partial^* B_s \cap R_s) + \max\{\sigma_A, \sigma_{AB}\} \mathcal{H}^1(A^{(1)} \cap \partial R_s) \\
&\qquad\qquad + \max\{\sigma_A, \sigma_B\}\big(|h_s^+(x_0 - s) - h_s^-(x_0 - s)| + |h_s^+(x_0 + s) - h_s^-(x_0 + s)|\big) \\
&\qquad\qquad + \Lambda m(s) + \Lambda |B_s \triangle B| \\
&\qquad \leq \sigma_B \mathcal{H}^1(\Gamma^B \cap R_s) + \sigma_B \mathcal{H}^1(\Gamma^{AB} \cap R_s) + 3\max\{\sigma_A, \sigma_B, \sigma_{AB}\} m'(s) + 3\Lambda m(s).
\end{aligned} \tag{4.85}$$

By using (4.70), (4.77), and the isoperimetric inequality, we estimate

$$\begin{aligned}
\sigma_A \mathcal{H}^1(\Gamma^A \cap R_s) &+ (\sigma_{AB} - \sigma_B) \mathcal{H}^1(\Gamma^{AB} \cap R_s) \\
&\geq \sigma_A \mathcal{H}^1(\Gamma^A \cap R_s) + \min\{\sigma_{AB} - \sigma_B, 0\} \mathcal{H}^1(\Gamma^A \cap R_s) \\
&\geq \delta \mathcal{H}^1(\Gamma^A \cap R_s) \geq \frac{\delta}{2} \mathcal{H}^1(\Gamma^{AB} \cap R_s) + \frac{\delta}{2} \mathcal{H}^1(\Gamma^A \cap R_s) \\
&= \frac{\delta}{2}\Big(\mathscr{P}(A \cap R_s) - \mathcal{H}^1(A^{(1)} \cap \partial R_s)\Big) \\
&\geq C\big(m(s)^{\frac{1}{2}} - m'(s)\big).
\end{aligned} \tag{4.86}$$

By inserting (4.86) into (4.85) we obtain the desired inequality (4.76) in the case where assumption (4.77) holds.

Assume now that (4.80) is in force, and thus A_s and B_s are defined as in (4.81). In this case, the quasi-minimality inequality for u yields, using also (4.74) and (4.75),

$$\begin{aligned}
\sigma_A \mathcal{H}^1(\Gamma^A \cap R_s) + \sigma_{AB} \mathcal{H}^1(\Gamma^{AB} \cap R_s) &\leq \sigma_B \mathcal{H}^1(\Gamma^A \cap R_s) + \sigma_{AB} \mathcal{H}^1(A^{(1)} \cap \partial R_s) + 2\Lambda m(s) \\
&\leq \sigma_B \mathcal{H}^1(\Gamma^A \cap R_s) + \sigma_{AB} m'(s) + 2\Lambda m(s).
\end{aligned} \tag{4.87}$$

By using (4.70), (4.80), and the isoperimetric inequality, we estimate

$$\begin{aligned}
(\sigma_A - \sigma_B)\mathcal{H}^1(\Gamma^A \cap R_s) + \sigma_{AB}\mathcal{H}^1(\Gamma^{AB} \cap R_s) \\
&\geq \min\{\sigma_A - \sigma_B, 0\}\mathcal{H}^1(\Gamma^{AB} \cap R_s) + \sigma_{AB}\mathcal{H}^1(\Gamma^{AB} \cap R_s) \\
&\geq \delta\mathcal{H}^1(\Gamma^{AB} \cap R_s) \geq \frac{\delta}{2}\mathcal{H}^1(\Gamma^{AB} \cap R_s) + \frac{\delta}{2}\mathcal{H}^1(\Gamma^A \cap R_s) \\
&= \frac{\delta}{2}\Big(\mathscr{P}(A \cap R_s) - \mathcal{H}^1(A^{(1)} \cap \partial R_s)\Big) \\
&\geq C\big(m(s)^{\frac{1}{2}} - m'(s)\big).
\end{aligned} \tag{4.88}$$

By inserting (4.88) into (4.87) we obtain the desired inequality (4.76) also in the case where assumption (4.80) holds.

Step 2. We conclude as follows. Let x_0 be as in the statement, and let $z_0 := (x_0, h(x_0)) \in \partial^* A \cup \partial^* B$. Assume $z_0 \in \partial^* B$. Let $r_0 > 0$ be as in Step 1, so that h is Lipschitz continuous in $(x_0 - r_0, x_0 + r_0)$ with Lipschitz constant ℓ. Let also $\varepsilon > 0$ be given by Step 1. Then, since $z_0 \in \partial^* B \cap \partial^* V$, it is possible to find $r \in (0, r_0)$ such that

$$|A \cap R_r(z_0)| < \varepsilon r^2.$$

From Step 1 we get that $|A \cap R_{r/2}| = 0$. This implies that in $R_{r/2}$ there are only the sets V and B. In particular, we also have $\partial^+ R_{r/2} \subset B^{(0)}$, $\partial^- R_{r/2} \subset B^{(1)}$ (recall (4.71)), and $B \cap R_{r/2}$ is the subgraph of a function of bounded variation.

Let now $E \subset R_{r/2}$ be any set of finite perimeter such that $E \triangle B \subset\subset R_{r/2}$, and let E^G be the set defined in (4.79). We can test the quasi-minimality inequality with the competitor obtained by replacing the phase $B \cap R_{r/2}$ by E^G, which is admissible since it satisfies the graph constraint. We therefore find

$$\sigma_B \mathscr{P}(B; R_{r/2}) \leq \sigma_B \mathscr{P}(E^G; R_{r/2}) + \Lambda|(E^G \triangle B) \cap R_{r/2}|. \tag{4.89}$$

We observe now that, similarly to (4.82), we have $\mathscr{P}(E^G; R_{r/2}) \leq \mathscr{P}(E; R_{r/2})$; moreover (since $B \cap R_{r/2} = (B \cap R_{r/2})^G$) we also have

$$\begin{aligned}
|(E^G \triangle B) \cap R_{r/2}| &= |E^G \triangle (B \cap R_{r/2})^G| \\
&= |E^G| + |(B \cap R_{r/2})^G| - 2|E^G \cap (B \cap R_{r/2})^G| \\
&\leq |E| + |B \cap R_{r/2}| - 2|E \cap (B \cap R_{r/2})| = |E \triangle (B \cap R_{r/2})|.
\end{aligned}$$

Then, inserting the previous inequalities inside (4.89) we find

$$\mathscr{P}(B; R_{r/2}) \leq \mathscr{P}(E; R_{r/2}) + \frac{\Lambda}{\sigma_B}|(E \triangle B) \cap R_{r/2}|$$

for every set of finite perimeter $E \subset R_{r/2}$ such that $E \triangle B \subset\subset R_{r/2}$.

This shows that B is a quasi-minimizer of the perimeter inside $R_{r/2}$ in the classical sense, that is *without* the graph constraint. Thus, the regularity of $\Gamma_h \cap R_{r/2} = \partial^* B \cap R_{r/2}$ follows from classical regularity results for quasi-minimizers of the perimeter (see, for instance, [22, Theorems 26.5 and 28.1]). This concludes the proof. □

By collecting all the previous statements, we obtain the properties listed in Theorem 1.1.

Proof of Theorem 1.1. The infiltration property (i) follows by Proposition 4.6 and Proposition 4.7. The Lipschitz regularity (ii) and the characterization of the singular set (iii) are proved in Proposition 4.10 and Remark 4.11. The internal regularity of the interface (iv) is discussed in Remark 4.12. The $C^{1,\alpha}$-regularity of the graph (v) is proved in Proposition 4.13. □

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